

# Quantum charges and spacetime topology: The emergence of new superselection sectors

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**Dedicated to Klaus Fredenhagen on the occasion of his sixtieth birthday**

## Abstract

In which is developed a new form of superselection sectors of topological origin. By that it is meant a new investigation that includes several extensions of the traditional framework of Doplicher, Haag and Roberts in local quantum theories. At first we generalize the notion of representations of nets of  $C^*$ -algebras, then we provide a brand new view on selection criteria by adopting one with a strong topological flavour. We prove that it is coherent with the older point of view, hence a clue to a genuine extension. In this light, we extend Roberts’ cohomological analysis to the case where 1-cocycles bear non trivial unitary representations of the fundamental group of the spacetime, equivalently of its Cauchy surface in case of global hyperbolicity. A crucial tool is a notion of group von Neumann algebras generated by the 1-cocycles evaluated on loops over fixed regions. One proves that these group von Neumann algebras are localized at the bounded region where loops start and end and to be factorial of finite type  $I$ . All that amounts to a new invariant, in a topological sense, which can be defined as the dimension of the factor. We prove that any 1-cocycle can be factorized into a part that contains only the charge content and another where only the topological information is stored. This second part resembles much what in literature are known as geometric phases. Indeed, by the very geometrical origin of the 1-cocycles that we discuss in the paper, they are essential tools in the theory of net bundles, and the topological part is related to their holonomy content. At the end we prove the existence of net representations.

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# 1 Introduction

A large class of effects in physics can be explained using the features and language of topology. Starting from those due to non-trivial topology of configuration spaces in classical and quantum physics, one also finds intriguing topological effects in general relativity and quantum field theory. To name just a few of the most interesting ones we recall the Ehrenberg-Siday-Aharonov-Bohm effect [21, 1] in quantum mechanics and its generalization named Pancharatnam-Berry phase [43, 7], or its classical counterpart called Hannay's angle [29], the Jahn-Teller effect [33], Wheeler's geons [58] and the Casimir effects [15].

The aim of the present paper is to trace the route for a rigorous attack to the dependence on topology of certain structures in quantum field theory. In particular, we aim at a complete model independent description, at least in a preliminary case where the topological effects under interest are those related to specific topological features of spacetimes. The language is that of local quantum theory [28, 9] (otherwise called algebraic quantum field theory) where, as it is well known, one finds the best understanding about the nature and properties of structural properties of quantum field theories. The proper setting is that related to the prominent case of superselection sectors where charged quantum numbers find their definition as attributes associated to unitary equivalence classes of representations of the net of local observable algebras satisfying certain selection conditions.

The traditional analysis of such selection criteria – and associated equivalence classes of representations thereof – goes mainly by the names of Doplicher, Haag, and Roberts [18]. They worked out the structure of charges localizable into bounded regions, whilst it is due to Buchholz and Fredenhagen [13] the study of charges that can be localized in unbounded regions, i.e. spacelike cones. All that was done for quantum field theories on Minkowski spacetime in dimensions  $d \geq 3$ . Other groups of researchers have been able to follow the same route in various directions, especially in the direction of conformal quantum field theory in two dimensions, and besides the crucial results of Fredenhagen, Rehren and Schroer [24], the main success was obtained by Kawahigashi and Longo in [34], where they have been able to completely classify theories with central charge less than one.

The authors of the present paper have recently put forward an analysis of the structure of superselection sectors [12] that provides a new perspective both by the adopted techniques and in the fact that superselection theory is now applicable to the larger setting of locally covariant quantum field theories [11]. The obtained results confirm that sectors of the kind that Doplicher, Haag and Roberts studied long ago find their most natural position in the locally covariant framework. In fact, we can associate with any 4-dimensional globally hyperbolic spacetime a unique, symmetric, tensorial  $C^*$ -category (that possesses conjugates in case of finite statistics) and that to any isometric embedding between such spacetimes the previous categories can be contravariantly related as to guarantee that charges are preserved under the embedding.

The local covariance of sectors comes from the analysis of 1-cocycles associated

with the Roberts' cohomology of posets [45, 46, 47, 48, 53] that carry a trivial unitary representation of the fundamental group of the spacetime.

It is natural to try to understand the kind of 1-cocycles that carry a non-trivial unitary representation and to see whether one can associate with them a different kind of superselection sector and charge, now attributable to the possible non-trivial topology of the spacetime. The main results of the following analysis show that this is indeed possible and fruitful.

We have now a clear relation between topological properties of a spacetime and structural properties of 1-cocycles carrying non-trivial representations of the fundamental group of the spacetime. The analysis that follows, however, is not yet casted into a locally covariant form, although our initial aim was into that direction. We will be working on a fixed, but otherwise arbitrary, 4-dimensional globally hyperbolic spacetime. We hope to return elsewhere to the locally covariant analysis, and consider this paper as the third one of the announced series in [12].

The main ingredients are the following: first, a generalization of the usual notion of representations of a net of local algebras, something that we termed “unitary net representations;” secondly, the association of this new notion with a 1-cocycle, in the sense of Roberts' cohomology of posets; thirdly, a new selection criterion that generalizes that of Doplicher, Haag and Roberts, for the sake of attributing a non-trivial dependence on the spacetime topology to the superselection sectors so defined; fourthly, one defines a von Neumann algebra which is the group algebra generated by a 1-cocycle evaluated on all loops over a fixed bounded region, as a starting and ending point, and proves that this algebra is localized, i.e. it is a subalgebra of the von Neumann algebra of the net that is localized in the chosen region. This last ingredient is the key structural element of the analysis that follows. It allows us to attribute to each 1-cocycle generating its own group von Neumann algebra a new invariant, called “topological dimension,” that resembles much, and has similar properties of a charge quantum number, and carries non-trivial informations about the topology of the spacetime.

Furthermore, we prove that any 1-cocycle can be splitted into a part that carry only information on the charge content of the sector and a part that carry the topological information of spacetime. This last resembles much what we cited at the beginning as geometric phases. For more on that see also Section 7, where we also prove the existence of net representations.

Abstract as they are, one would like to have concrete examples of constructions of this non-Abelian kind of geometric phases. A recent work, done by one of the authors in collaboration with Franceschini and Moretti [8], provides a first explicit example of a 1-cocycle induced by the non-trivial topology of spacetime in the simple case of massive bosonic quantum field theory on the 2-dimensional Einstein cylinder.

A further glance at models may indicates other situations where our analysis may apply. For instance, in cosmology one looks for visible effects of the non-trivial topology of spacetimes by searching for additional images in the sky of the same galaxy, due to the possible presence of a cosmic string. Besides that, we mention also that there is a large class of physically meaningful multiply connected spacetimes. These spacetimes

are a class of Friedmann-Lamaître models, solutions of the Einstein equations which are used as cosmological models (see [36, 55]).

We finally point the reader to a recent interesting paper by Morchio and Strocchi [40] where, in the case of quantum mechanics on manifolds, they describe a classification of topological effects in close analogy to our results. Also, papers by Döbner et al. [17] and Landsman [37], have a similar flavour.

The paper has been structured in such a way to maintain a decent ratio between size and completeness, hence many results are presented into the simplest form that we could think of. We refer the reader to [38, 19, 53, 12] for a deeper introduction to some of the mathematical notions that we use.

## 2 The category of net representations

We introduce the notion of net representations for nets of  $C^*$ -algebras. We analyze in particular the class of unitary net representations pointing out their topological content. The importance of this new notion resides in the fact that a new class of superselection sectors induced by the topology of spacetimes is described in terms of net representations (see next sections). We shall use the tool of cohomology of posets to make explicit the topological information carried by net representations. Within this section we shall also discuss, very briefly, preliminary informations on the simplicial set associated with a poset and the first degree of its cohomology. Details can be found in [48, 53, 12, 50].

Let  $K$  be a poset with order relation  $\leq$ . We consider the simplicial set  $\Sigma_*(K)$  of *singular simplices* associated with  $K$ . We use the standard symbols  $\partial_i$  and  $\sigma_i$  to denote the face and degeneracy maps, and denote the compositions  $\partial_i\partial_j$ ,  $\sigma_i\sigma_j$  respectively by  $\partial_{ij}$  and  $\sigma_{ij}$ . We pass now to a brief definition of the set  $\Sigma_n(K)$  of  $n$ -simplices. A 0-simplex is just an element of  $K$ . Inductively, for  $n \geq 1$ , and  $n$ -simplex  $x$  is formed by  $n + 1$ ,  $(n - 1)$ -simplices  $\partial_0x, \dots, \partial_nx$  and by an element of the poset  $|x|$ , called the *support* of  $x$ , such that  $|\partial_ix| \leq |x|$  for  $i = 0, \dots, n$ . We shall denote 0-simplices either by  $a$  or by  $o$ , 1-simplices by  $b$ , and 2-simplices by  $c$ . Given a 1-simplex  $b$  the *reverse*  $\bar{b}$  is the 1-simplex having the same support as  $b$  and such that  $\partial_0\bar{b} = \partial_1b$ ,  $\partial_1\bar{b} = \partial_0b$ .

Composing 1-simplices one gets paths. A *path*  $p$  is a finite ordered set of 1-simplices  $b_n * \dots * b_1$  satisfying the relations  $\partial_0b_{i-1} = \partial_1b_i$  for  $i = 2, \dots, n$ . We define the 0-simplices  $\partial_1p \doteq \partial_1b_1$  and  $\partial_0p \doteq \partial_0b_n$  and call them, respectively, the *starting* and the *ending* point of  $p$ . The *support* of a path is defined as the union of the supports of the 1-simplices by which it is composed. By  $p : a \rightarrow \tilde{a}$  we mean a path starting from  $a$  and ending at  $\tilde{a}$ . The *reverse* of  $p$  is the path  $\bar{p} : \tilde{a} \rightarrow a$  defined by  $\bar{p} \doteq \bar{b}_1 * \dots * \bar{b}_n$ . If  $q$  is a path from  $\tilde{a}$  to  $\hat{a}$ , then we can define, in an obvious way, the composition  $q * p : a \rightarrow \hat{a}$ . The poset  $K$  is said to be *pathwise connected* whenever for any pair  $a, \tilde{a}$  of 0-simplices there is a path from  $a$  to  $\tilde{a}$ .

Let  $p = b_n * \dots * b_1$  be a path. An *elementary deformation* of  $p$  consists in replacing a 1-simplex  $\partial_1c$  of the path by the pair  $\partial_0c * \partial_2c$ , where  $c \in \Sigma_2(K)$ , or conversely in replacing a consecutive pair  $\partial_0c * \partial_2c$  by a single 1-simplex  $\partial_1c$ . Two paths with the same

endpoints are *homotopic* if they can be obtained from one another by a finite sequence of elementary deformations. Homotopy defines an equivalence relation  $\sim$  on the set of paths with the same endpoints which is compatible with reverse and composition. The *first homotopy group* of the poset  $\pi_1(K, a)$ , with base point  $a$ , is the quotient of the set  $\text{Loops}_K(a)$  of all paths  $p : a \rightarrow a$  in  $K$  with respect to the homotopy equivalence relation. If  $K$  is pathwise connected the first homotopy group does not depend, up to isomorphism, on the base point. The isomorphism class is the fundamental group  $\pi_1(K)$  of the poset and we will say that  $K$  is simply connected whenever the fundamental group is trivial.

We conclude this introductory part with two remarks. First, we recall that if  $K$  is upward directed, namely if for any  $a_1, a_2 \in K$  there is  $a_3 \in K$  with  $a_1, a_2 \leq a_3$ , then it is pathwise and simply connected. Secondly, let  $M$  be a topological space and consider a basis of its topology formed by open arcwise and simply-connected open sets of  $M$ . If  $K$  is a poset formed by the elements of this basis with the inclusion order relation  $\subseteq$ , then  $\pi_1(M) = \pi_1(K)$ .

We now turn to the definition of net representations. From now on we fix a poset  $K$  and assume that it is pathwise connected. A *net* of  $C^*$ -algebras  $\mathcal{A}_K$  on a poset  $K$  is given by the following data: there is mapping  $a \mapsto \mathcal{A}(a)$  from  $K$  to unital  $C^*$ -algebras; for any pair  $\tilde{a}, a \in K$  with  $\tilde{a} \leq a$ , there is an injective  $*$ -morphism  $j_{a\tilde{a}} : \mathcal{A}(\tilde{a}) \rightarrow \mathcal{A}(a)$ . The morphisms  $j_{a\tilde{a}}$  are called *inclusion morphisms*. These morphisms are required to satisfy the following coherence property

$$j_{a\tilde{a}} j_{\tilde{a}\hat{a}} = j_{a\hat{a}} , \quad \hat{a} \leq \tilde{a} \leq a . \quad (2.1)$$

A *net representation* of  $\mathcal{A}_K$  is a pair  $\{\pi, \psi\}$ , where  $\pi$  denotes a function that associates a representation  $\pi_a$  of  $\mathcal{A}(a)$  on a Hilbert space  $\mathcal{H}_a^\pi$  with any  $a \in K$ ;  $\psi$  denotes a function that associates an injective linear operator  $\psi_{a\tilde{a}} : \mathcal{H}_{\tilde{a}}^\pi \rightarrow \mathcal{H}_a^\pi$  with any pair  $a, \tilde{a} \in K$ , with  $\tilde{a} \leq a$ . The functions  $\pi$  and  $\psi$  are required to satisfy the following relations

$$\psi_{a\tilde{a}} \pi_{\tilde{a}} = \pi_a j_{a\tilde{a}} \psi_{a\tilde{a}} , \quad \tilde{a} \leq a , \quad \text{and} \quad \psi_{a\tilde{a}} \psi_{\tilde{a}\hat{a}} = \psi_{a\hat{a}} , \quad \hat{a} \leq \tilde{a} \leq a . \quad (2.2)$$

An *intertwiner* from  $\{\pi, \psi\}$  to  $\{\rho, \phi\}$  is a function  $T$  associating a bounded linear operator  $T_a : \mathcal{H}_a^\pi \rightarrow \mathcal{H}_a^\rho$  with any  $a \in K$ , and satisfying the relations

$$T_a \pi_a = \rho_a T_a , \quad \text{and} \quad T_a \psi_{a\tilde{a}} = \phi_{a\tilde{a}} T_{\tilde{a}} , \quad \tilde{a} \leq a . \quad (2.3)$$

We denote the set of intertwiners from  $\{\pi, \psi\}$  to  $\{\rho, \phi\}$  by the symbol  $(\{\pi, \psi\}, \{\rho, \phi\})$ , and say that the net representations are *unitarily equivalent* if they have a unitary intertwiner  $T$ , that is,  $T_a$  is a unitary operator for any  $a$ .

The definition of net representation is suggested by the theory of bundles over posets [50]. There is, in fact, an underlying structure of a net bundle and, as we shall point out, some results on net representations are analogous to those of net bundles. The contact point resides in the defining properties of the function  $\psi$ , which are the same as the net structure of a net bundle. However, net representations already appeared

in the literature of algebraic quantum field theory, although not in this general form. They have been considered by Buchholz, Haag and Roberts in an unpublished paper.<sup>1</sup> Fredenhagen and Haag encountered this class of representations in the reconstruction of a theory from its germs [23]. An argument used in that paper allows us to show how net representations arise. We call a *net state* a function  $\omega$  associating a state  $\omega_a$  of  $\mathcal{A}(a)$  with any  $a \in K$ , and which is compatible with the inclusion morphisms, i.e.,

$$\omega_a j_{a\tilde{a}} = \omega_{\tilde{a}} , \quad \tilde{a} \leq a . \quad (2.4)$$

Given a net state  $\omega$ , denote the GNS-construction of  $\omega_a$  by  $\{\pi_a, \mathcal{H}_a, \Omega_a\}$ , and define

$$\psi_{a\tilde{a}} \pi_{\tilde{a}}(A) \Omega_{\tilde{a}} \doteq \pi_a(j_{a\tilde{a}}(A)) \Omega_a , \quad \tilde{a} \leq a . \quad (2.5)$$

By using (2.4), we have that  $\psi_{a\tilde{a}} : \mathcal{H}_{\tilde{a}} \rightarrow \mathcal{H}_a$  is an isometry. Furthermore, it is a routine calculation to check that  $\psi_{a\tilde{a}} \pi_{\tilde{a}} = \pi_a j_{a\tilde{a}} \psi_{a\tilde{a}}$  for any  $\tilde{a} \leq a$ . Finally observe that, by the defining equation of  $\psi$  and by (2.1) we have

$$\begin{aligned} \psi_{a\tilde{a}} \psi_{\tilde{a}\hat{a}} \pi_{\hat{a}}(A) \Omega_{\hat{a}} &= \psi_{a\tilde{a}} \pi_{\tilde{a}}(j_{a\tilde{a}}(A)) \Omega_{\tilde{a}} = \pi_a(j_{a\tilde{a}} j_{\tilde{a}\hat{a}}(A)) \Omega_a \\ &= \pi_a(j_{a\hat{a}}(A)) \Omega_a = \psi_{a\hat{a}} \pi_{\hat{a}}(A) \Omega_{\hat{a}} \end{aligned}$$

for any  $\hat{a} \leq \tilde{a} \leq a$  and  $A \in \mathcal{A}(\hat{a})$ , and this implies that  $\psi_{a\tilde{a}} \psi_{\tilde{a}\hat{a}} = \psi_{a\hat{a}}$ . So the pair  $\{\pi, \psi\}$  is a net representation.

In the present paper we are interested in unitary net representations. A net representation  $\{\pi, \psi\}$  is said to be *unitary* whenever  $\psi_{a\tilde{a}}$  is a unitary operator for any  $\tilde{a} \leq a$ . An interesting feature is that, since  $K$  is pathwise connected, any unitary net representation is equivalent to a unitary net representation on a fixed Hilbert space. The argument is the same as that used to prove that any net bundle has a standard fibre (see [50, Proposition 4.5]). Since the operators  $\psi_{a\tilde{a}}$  are unitary, by a standard argument, one defines a unitary operator  $V_a$  from a fixed Hilbert space  $\mathcal{H}$  to  $\mathcal{H}_a^\pi$  for any  $a \in K$ . Afterwards one defines

$$\rho_a \doteq V_a^* \pi_a V_a , \quad a \in K , \quad \phi_{a\tilde{a}} \doteq V_a^* \psi_{a\tilde{a}} V_{\tilde{a}} , \quad \tilde{a} \leq a .$$

The pair  $\{\rho, \psi\}$  is a unitary net representation on the Hilbert space  $\mathcal{H}$ ; the function  $V : K \ni a \rightarrow V_a$  defines a unitary intertwiner from  $\{\rho, \phi\}$  to  $\{\pi, \psi\}$ . From now on we will consider only unitary net representations in a fixed Hilbert space.

We denote by  $\text{Rep}^{net}(\mathcal{A})$  the set of unitary net representations of  $\mathcal{A}$  and by the same symbol the category having unitary net representations of  $\mathcal{A}$  as objects and the corresponding intertwiners as arrows. We call this one the category of *unitary net representations* of  $\mathcal{A}$ . If the target of an arrow  $T$  is equal to the source of  $S$ , the composition  $S \cdot T$  is defined by  $(S \cdot T)_a = S_a T_a$  for any  $a \in K$ . The identity arrow  $1_{\{\pi, \psi\}}$  is  $(1_{\{\pi, \psi\}})_a = 1_{\mathcal{H}^\pi}$  for any  $a$ . Furthermore,  $\text{Rep}^{net}(\mathcal{A})$  is a  $\text{C}^*$ -category. The adjoint  $*$  is defined as the identity on objects, while on arrows  $T$  it is defined as  $(T^*)_a = T_a^*$  for any  $a$ . Finally, given an arrow  $T$ , then  $\|T\| \doteq \sup_{a \in K} \|T_a\|$  is a norm which makes  $\text{Rep}^{net}(\mathcal{A})$  a  $\text{C}^*$ -category. Note that by (2.3), since  $K$  is pathwise connected,  $\|T_a\|$  is constant.

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<sup>1</sup>Private communication by J. E. Roberts.

We now use the cohomology of posets to make explicit the topological content of unitary net representations. We give only a brief introduction of this topic and refer the reader to the papers quoted at the beginning for details. Consider a Hilbert space  $\mathcal{H}$ . A 1-cocycle  $z$  of the poset  $K$ , with values in  $\mathfrak{B}(\mathcal{H})$ , is a function  $z : \Sigma_1(K) \ni b \rightarrow z(b) \in \mathfrak{B}(\mathcal{H})$  of unitary operators of  $\mathcal{H}$  satisfying the equation

$$z(\partial_0 c) z(\partial_2 c) = z(\partial_1 c) , \quad c \in \Sigma_2(K) . \quad (2.6)$$

The *trivial* 1-cocycle  $\iota$  is defined by  $\iota(b) = \mathbb{1}_{\mathcal{H}}$  for any 1-simplex  $b$ . A 1-cocycle is a 1-coboundary if there is a function  $v : \Sigma_0(K) \ni a \rightarrow v_a \in \mathfrak{B}(\mathcal{H})$  of unitary operators such that  $z(b) = v_{\partial_0 b}^* v_{\partial_1 b}$  for any 1-simplex  $b$ . We denote the set of 1-cocycles by  $Z^1(K, \mathfrak{B}(\mathcal{H}))$ . Given a pair  $z, \tilde{z}$  of 1-cocycles an *intertwiner*  $t$  from  $z$  to  $\tilde{z}$  is a function  $t : \Sigma_0(K) \ni a \rightarrow t_a \in \mathfrak{B}(\mathcal{H})$  satisfying

$$t_{\partial_0 b} z(b) = \tilde{z}(b) t_{\partial_1 b} , \quad b \in \Sigma_1(K) . \quad (2.7)$$

We denote the set of intertwiners from  $z$  to  $\tilde{z}$  by  $(z, \tilde{z})$ . The category of 1-cocycles is the category whose objects are 1-cocycles and whose arrows are the corresponding set of intertwiners. We denote this category by the same symbol  $Z^1(K, \mathfrak{B}(\mathcal{H}))$  as that used to denote the corresponding set of objects. This is a  $C^*$ -category: composition of arrows and the adjoint are defined in the same way as in  $\text{Rep}^{net}(\mathcal{A}_K)$  (see [47, 53] for details). Two 1-cocycles  $z, \tilde{z}$  are unitarily *equivalent* if there exists a unitary arrow  $t \in (z, \tilde{z})$ . Observe that any 1-coboundary is unitarily equivalent to the trivial 1-cocycle  $\iota$ .

We extend a 1-cocycle  $z$  from 1-simplices to paths by setting

$$z(p) \doteq z(b_n) \cdots z(b_2) z(b_1) , \quad p = b_n * \cdots * b_1 .$$

It is easily seen that  $z(\bar{p}) = z(p)^*$  for any path  $p$ , and if  $p$  and  $q$  are homotopic then  $z(p) = z(q)$  (*homotopic invariance*). These properties imply that any 1-cocycle defines a unitary representation, denoted by  $z$ , in  $\mathcal{H}$  of the fundamental group of the poset. Using this result the topological content of a unitary net representations is easily analyzed. Indeed, given a unitary net representation  $\{\pi, \psi\}$ , define

$$\zeta^\pi(b) \doteq \psi_{|b|, \partial_0 b}^* \psi_{|b|, \partial_1 b} , \quad b \in \Sigma_1(K) . \quad (2.8)$$

One can check that  $\zeta^\pi$  is a 1-cocycle of  $K$  with values in the group of unitary operators of  $\mathcal{H}_\pi$  ([50]). This 1-cocycle defines a representation of the fundamental group of the poset. Thus, we say that a net representation  $\{\pi, \phi\}$  is *topologically trivial* whenever  $\zeta^\pi$  is a 1-coboundary. Thus, if  $K$  is simply connected then any unitary net representation is topologically trivial.

**Lemma 2.1.** *Assume that  $\{\pi, \psi\}$  and  $\{\rho, \phi\}$  are unitarily equivalent. Then the corresponding 1-cocycles  $\zeta^\pi$  and  $\zeta^\rho$  are equivalent.*



*Proof.* Let  $W \in (\{\pi, \psi\}, \{\rho, \phi\})$  be unitary. Then

$$\begin{aligned} W_{\partial_0 b} \zeta^\pi(b) &= W_{\partial_0 b} \psi_{|b|, \partial_0 b}^* \psi_{|b|, \partial_1 b} = (\psi_{|b|, \partial_0 b} W_{\partial_0 b}^*)^* \psi_{|b|, \partial_1 b} \\ &= (W_{|b|}^* \phi_{|b|, \partial_0 b})^* \psi_{|b|, \partial_1 b} = \phi_{|b|, \partial_0 b}^* W_{|b|} \psi_{|b|, \partial_1 b} \\ &= \zeta^\rho(b) W_{\partial_1 b} . \end{aligned}$$

Hence  $W \in (\zeta^\pi, \zeta^\rho)$  and this proves the assertion.  $\square$

Thus, equivalent unitary net representation have the same topological content (the converse, in general, does not hold as we will see at the end of this section).

**Lemma 2.2.** *Let  $\{\pi, \psi\}$  be a topologically trivial unitary net representation. Then  $\{\pi, \psi\}$  is equivalent to a unitary net representation of the form  $\{\rho, \mathbb{1}\}$ .*

*Proof.* Since  $\zeta^\pi$  is a 1-coboundary there exists a family of unitary operators  $W_a : \mathcal{H} \rightarrow \mathcal{H}$  such that  $\zeta^\pi(b) = W_{\partial_0 b}^* W_{\partial_1 b}$ . For any 0-simplex  $a$ , define  $\rho_a(A) \doteq W_a \pi_a(A) W_a^*$ , with  $A \in \mathcal{A}(a)$ . It is clear that  $W_a \pi_a = \rho_a W_a$ . Moreover,  $W_a \psi_{a, \tilde{a}} = W^a \zeta^\pi(a; a, \tilde{a}) = W_{\tilde{a}}$ , where  $(a; a, \tilde{a})$  is the 1-simplex whose support is  $a$ , and whose 0- and 1-face are respectively  $a$  and  $\tilde{a}$ . This completes the proof.  $\square$

A unitary net representation can be easily defined starting from a representation  $\chi$  of the fundamental group of  $K$  with values in the complex number  $\mathbb{C}$ . It is shown in [53] that there is a 1-cocycle, associated with  $\chi$ . We maintain the symbol  $\chi$  to denote this 1-cocycle and define (see the previous proof for notation)

$$\psi_{aa}^\chi \doteq \chi(\tilde{a}; \tilde{a}, a) , \quad a \leq \tilde{a} . \quad (2.9)$$

Consider now a topologically trivial net representation  $\{\pi, \mathbb{1}\}$ . Since  $\psi^\chi$  takes values in the complex numbers, the pair  $\{\pi, \psi^\chi\}$  is a unitary net representation (note that  $\{\pi, \psi^\chi\}$  and  $\{\pi, \mathbb{1}\}$  are not equivalent because of Lemma 2.1). So, if the fundamental group of the poset is Abelian, then there are topologically non-trivial unitary net representations (clearly if the net is not trivial). For the non-Abelian case, topologically non-trivial examples, which are of interest for the theory of superselection sectors, will be given in Section 7. Finally, note that the above example shows that there are non equivalent unitary net representations whose 1-cocycles are equivalent.

We conclude this section with some observations.

(1) A net is nothing but a *precosheaf*. By reverting the arrows, the results of this section apply also to the duals, i.e. to presheaves, either of  $C^*$ -algebras or of groups. Briefly, given a presheaf over a poset, if we have a presheaf representation on a Hilbert space whose *restriction* morphisms are implemented by unitary operators satisfying relations corresponding to (2.2), then the presheaf representation carries a representation of the fundamental group of the poset. Proceeding as done for net representations, one gets a 1-cocycle of the dual poset  $K^\circ$  (the poset having the same elements as  $K$  with

opposite order relation). However, as shown in [50], the fundamental group of  $K$  is isomorphic to that of  $K^\circ$ .

(2) The notion of representation of a net of  $C^*$ -algebras, usually considered in the applications to quantum fields theory (see for instance [30, 18, 13, 26, 53, 12]), corresponds, in our framework, to a topologically trivial net representation. To see this note that, in the cited papers, by a representation of a net  $\mathcal{A}_K$  it is meant a function  $\pi$  associating to any  $a \in K$  a representation of  $\mathcal{A}(a)$  in a fixed Hilbert space  $\mathcal{H}^\pi$ , and such that  $\pi_a J_{a\tilde{a}} = \pi_{\tilde{a}}$  for any  $\tilde{a} \leq a$ . An intertwiner  $S$  from a representation  $\pi$  to a representation  $\rho$ , is a bounded linear operator  $S : \mathcal{H}^\pi \rightarrow \mathcal{H}^\rho$  such that  $S\pi_a = \rho_a S$  for any  $a \in K$ . Now, it is clear that a representation is a unitary net representation of the form  $\{\pi, \mathbb{1}\}$ . Moreover, if  $T$  is an intertwiner from  $\{\pi, \mathbb{1}\}$  to  $\{\rho, \mathbb{1}\}$ , then, according to (2.3), we have  $T_a = T_{\tilde{a}}$  for any  $\tilde{a} \leq a$ . Since  $K$  is pathwise connected, we have  $T_a = T_{\hat{a}}$  for any pair  $a, \hat{a}$ . So  $T$  is constant. This shows that our definition is indeed a generalization, and in particular that the category of representations is equivalent to the full subcategory of  $\text{Rep}^{net}(\mathcal{A}_K)$  whose net representations are topologically trivial. We will denote this category by  $\text{Rep}_t^{net}(\mathcal{A}_K)$ .

(3) Carpi, Longo and Kawahigashi have considered representations of net over the covering spaces of  $S^1$  [14]. We think that unitary net representations are of the same nature of those considered by them. We prefer however not to explore this topic in the present paper.

### 3 Charged sectors induced by topology

In [18], Doplicher, Haag and Roberts were able to select a class of superselection sectors of the observable net which manifest a covariant charge structure. These sectors, known as DHR-sectors, are representations of the observable net, in Minkowski spacetime, which are sharp excitations of the vacuum representation. This feature has been used to extend the notion of a DHR-sector to curved spacetimes [26]. In 4-dimensional globally hyperbolic spacetimes, DHR-sectors have a charge structure [26, 48, 53], which is generally covariant [12]. It is now clear that these sectors are not induced by the topology of spacetimes since they are associated with representations of the observable net which are, in the terminology introduced in the present paper, topologically trivial. In this section, by taking into account unitary net representations, we try to check whether they lead to genuine superselection structures. We start by discussing aspects of the causal structure of globally hyperbolic spacetimes, introduce the observable net, and the reference unitary net representation of the theory. The definition of sharply localized unitary net representations concludes the section. For simplicity, from now on by a net representation we will always mean a unitary net representation.

#### 3.1 The observable net

We start by discussing some aspects of the causal structure of globally hyperbolic spacetimes. The focus is on the set of diamonds, the class of regions that we will use as

indices of the observable net. Standard properties of globally hyperbolic spacetimes can be found [22, 4, 42, 54, 57]. Some advanced aspects can be found in [5, 6, 39].

Consider a 4-dimensional connected globally hyperbolic spacetime  $M$ . The *causal disjointness* relation is a symmetric binary relation  $\perp$  defined on subsets of  $M$  as follows:

$$o \perp \tilde{o} \iff o \subseteq M \setminus J(o) , \quad (3.1)$$

where  $J$  denotes the causal set of  $o$ . The causal complement of a set  $o$  is the open set  $o^\perp \doteq M \setminus cl(J(o))$ . An open set  $o$  is causally complete whenever  $o = o^{\perp\perp}$ . Now, the observable net over the spacetime  $M$  is a correspondence from open sets of the spacetime to the observables localized within these regions. In general not all the open sets are suited for this scope, since one needs a family of sets which fits very well both the topological and the causal properties of  $M$ . Moreover, additional conditions are imposed by the study of the observable nets derived from models of quantum fields [56, 52]. A family of sets that satisfies all these requirements is the set  $K(M)$  of *diamonds* of  $M$  [53]. A diamond of  $M$  is a subset  $o$  of  $M$  such that there is a spacelike Cauchy surface  $\mathcal{C}$ , a chart  $(U, \phi)$  of  $\mathcal{C}$ , and an open ball  $B$  of  $\mathbb{R}^3$  such that

$$o = D(\phi^{-1}(B)) , \quad cl(B) \subset \phi(U) \subset \mathbb{R}^3 , \quad (3.2)$$

where  $D(\phi^{-1}(B))$  is the domain of dependence of  $\phi^{-1}(B)$ , and such that  $cl(o)$  is *compact*. We will say that  $o$  is *based* on  $\mathcal{C}$  and call  $\phi^{-1}(B)$  the *base* of  $o$ . It turns out that, a diamond is an open, relatively compact, connected and simply connected subset of  $M$ . Any diamond  $o$  is causally complete, and the causal complement  $o^\perp$  is connected. The set of diamonds  $K(M)$  of  $M$  is a base for the topology of  $M$ . Some technical properties of diamonds are shown in Appendix B. Notice that our present definition of diamonds differs, by the request of compactness of the closure, from the original one in [53]. The results provided there and in [12] do not change after restriction to this smaller class.

We call a *subspacetime* of  $M$  any globally hyperbolic open connected subset of  $M$ . Diamonds and their causal complements are examples of subspacetimes of  $M$ . Another example is the *causal puncture*  $x^\perp$  in a point  $x \in M$ . This is nothing but the causal complement of the point  $x$ . Now, it is an easy consequence [12] of a powerful result on the deformation of Cauchy surfaces [6], that

$$K(N) = \{o \in K(M) : cl(o) \subset N\} , \quad (3.3)$$

for any subspacetime  $N$  of  $M$ .

We now move toward the definition of the observable net, and consider the poset formed by the set of diamonds of  $M$  ordered under inclusion  $\subseteq$ . Some topological information of the spacetime can be deduced from the poset  $K(M)$ . First of all, the poset  $K(M)$  is pathwise connected since  $M$  is connected. Secondly, the first homotopy group of  $K(M)$  is isomorphic to the first homotopy group of  $M$ . Furthermore, recall that if a poset is upward directed, then it is simply connected (see Section 2). Thus

$K(M)$  is not upward directed, when  $M$  is not simply connected. The same happens when  $M$  has compact Cauchy surfaces.

The observable net in the Minkowski spacetime is defined according to the Haag-Kastler axioms [30] (see also [28]). A generalization to a 4-dimensional globally hyperbolic spacetime  $M$  has been provided in [26]. The *observable net*  $\mathcal{A}_{K(M)}$  is defined as a correspondence

$$o \mapsto \mathcal{A}(o) , \quad (3.4)$$

associating with any diamond  $o$  of  $M$  a unital  $C^*$ -algebra  $\mathcal{A}(o)$  representing the algebra generated by *all* the observables localized within  $o$ , and satisfying the *isotony relation*

$$o \subseteq \tilde{o} \Rightarrow \mathcal{A}(o) \subseteq \mathcal{A}(\tilde{o}) . \quad (3.5)$$

Isotony implies that the observable net is a net of  $C^*$ -algebras over the poset  $K(M)$ . Now, the Haag-Kastler axioms include the Einstein's causality principle, saying that observables localized in causally disjoint (spacelike separated) regions must commute. However, when the indices of the observable net is a non-upward directed poset this principle cannot be fully implemented. Recall that  $K(M)$  fails to be upward directed when the spacetime is not simply connected or when it has compact Cauchy surfaces. Following [26], we restore this principle to the level of net representations of the observable net. A net representation  $\{\pi, \psi\}$  of  $\mathcal{A}_{K(M)}$  is said to be *causal* whenever

$$o \perp \tilde{o} \Rightarrow \pi_o(\mathcal{A}(o)) \subseteq \pi_{\tilde{o}}(\mathcal{A}(\tilde{o}))' , \quad (3.6)$$

where the prime stands for the commutant of the algebra.

### 3.2 Sharply localized net representations: a selection criterion

We start by introducing the reference net representation. This net representation, that turns out to be a DHR-like representation because of the request of topological triviality, shall play for the theory the same rôle as the vacuum one in Minkowski spacetime. We conclude by giving the definition of net representations which are a sharp excitation of the reference one.

As a *reference* net representation of the observable net we consider a faithful, *causal and topologically trivial* net representation in an infinite separable complex Hilbert space  $\mathcal{H}_0$ . Thus, according to Lemma 2.2, we take a net representation of  $\mathcal{A}_{K(M)}$  of the form  $\{\iota, \mathbb{1}\}$ . Moreover, let  $\mathcal{R}_{K(M)}$  be the net of von Neumann algebras

$$o \mapsto \mathcal{R}(o) ,$$

where  $\mathcal{R}(o) \doteq \iota_o(\mathcal{A}(o))''$ , that is, the observable net in the reference representation. Note that because of causality if  $o \perp \tilde{o}$  then  $\mathcal{R}(o) \subseteq \mathcal{R}(\tilde{o})'$ . Then, we require that  $\mathcal{R}_{K(M)}$  satisfies the following properties.

*Irreducibility* :  $\mathbb{C} \mathbb{1} = \cap \{\mathcal{R}(o)' \mid o \in K(M)\}$ ;

*Outer regularity* :  $\mathcal{R}(o) = \cap\{\mathcal{R}(\tilde{o}) \mid cl(o) \subset \tilde{o}\};$

*Borchers property* : given  $o \in K(M)$  and a non-zero orthogonal projection  $E \in \mathcal{R}(o)$ , for any  $\tilde{o}$  with  $cl(o) \subset \tilde{o}$  there exists an isometry  $V \in \mathcal{R}(\tilde{o})$  such that  $VV^* = E$ ;

*Punctured Haag duality* : given a point  $x \in M$  there holds

$$\mathcal{R}(o) = \cap\{\mathcal{R}(\tilde{o})' : \tilde{o} \in K(M), \tilde{o} \perp o, cl(\tilde{o}) \perp \{x\}\},$$

for any  $o \in K(M)$  with  $cl(o) \perp \{x\}$ .

Apart from the explicit request of topological triviality and outer regularity, the reference representation is defined in the same way as in [53]. Outer regularity, in particular, enters the theory only at one point, namely at the equivalence between sharply localized representations and net cohomology (see Appendix A). We stress that physically meaningful examples of representations satisfying the defining properties of the reference representation are the representations of a free scalar field which satisfy the microlocal spectrum condition [56, 52], a generalization to globally hyperbolic spacetimes of the spectrum condition [10, 44].

As a consequence of the above assumptions (see [53]) the net  $\mathcal{R}_{K(M)}$  satisfies Haag duality, i.e.,  $\mathcal{R}(a) = \cap\{\mathcal{R}(\tilde{a})' : \tilde{a} \perp a\}$ , for any  $a \in K(M)$ ; and it is locally definite, i.e.,  $\mathbb{C}\mathbb{1} = \cap\{\mathcal{R}(o) : x \in o\}$  for any  $x \in M$ . Punctured Haag duality can be better understood by looking at the restriction of the theory to the causal punctures of the spacetime (see Section 3.1). Let  $\mathcal{R}_{K(x^\perp)}$  be the net obtained by restricting  $\mathcal{R}_{K(M)}$  to the set of diamonds  $K(x^\perp)$  with  $x \in M$ . Then  $\mathcal{R}_{K(x^\perp)}$  is an irreducible net satisfying Haag duality. We recall that the restriction to the causal punctures was the key idea for the understanding of the charge structure of DHR-sectors on globally hyperbolic spacetimes, mainly because the point  $x$  plays for the set  $K(x^\perp)$  the same rôle as the spacelike infinity in Minkowski spacetime.

The purpose now is to generalize the criterion, used in [26, 53] to select DHR-sectors, to net representations. DHR-sectors are topologically trivial net representations  $\{\pi, \mathbb{1}\}$  of  $\mathcal{A}_{K(M)}$  which are a sharp excitation of the reference, in symbols

$$\pi \upharpoonright o^\perp \cong \iota \upharpoonright o^\perp, \quad (3.7)$$

for any  $o \in K(M)$ . This means that for any  $o$  there is a unitary operator  $U^o : \mathcal{H}^\pi \rightarrow \mathcal{H}$  such that  $U^o \pi_a = \iota_a U^o$  for any  $a \perp o$ . This definition *does not work* if one takes into account net representations which are not topologically trivial. In fact assume that  $\{\pi, \psi\}$  is topologically non-trivial. If  $\{\pi, \psi\}$  were equivalent to  $\{\iota, \mathbb{1}\}$  on  $o^\perp$ , by Lemma 2.1 the 1-cocycle  $\zeta^\pi$  would be trivial on  $o^\perp$  and this leads to a contradiction. Indeed, let  $\ell$  be a loop of  $K(M)$  over a 0-simplex whose closure is contained in  $o^\perp$ . By Corollary B.7  $\ell$  is homotopic to a loop  $\ell'$  whose support has closure contained in  $o^\perp$ . Then by homotopic invariance of 1-cocycles  $\zeta^\pi(\ell) = \zeta^\pi(\ell') = 0$ . Hence  $\zeta^\pi$  should be trivial on  $K(M)$ .

This observation suggests how to modify (3.7): we shall require that the above criterion is satisfied only in restriction to simply connected subspacetimes of  $M$  (see Section 3.1). To be precise, we say that a causal net representation  $\{\pi, \psi\}$  is a *sharp excitation* of the reference one, if for any  $o \in K(M)$  and for any *simply connected* subspacetime  $N$  of  $M$ , such that  $cl(o) \subset N$ , there holds

$$\{\pi, \psi\} \upharpoonright o^\perp \cap N \cong \{\iota, \mathbb{1}\} \upharpoonright o^\perp \cap N. \quad (3.8)$$

This amounts to saying that there is a family  $W^{No} \doteq \{W_a^{No} : cl(a) \subset N, a \perp o\}$  of unitary operators from  $\mathcal{H}^\pi$  to  $\mathcal{H}_0$  such that

1.  $W_a^{No} \pi_a = \iota_a W_a^{No}$  ;
2.  $W_a^{No} \psi_{a\tilde{a}} = W_{\tilde{a}}^{No}$  for any  $\tilde{a} \subseteq a$  ;
3.  $W^{No} = W^{N'o}$  for any simply connected subspacetime  $N'$  with  $N \subseteq N'$ .

These three equations represent the *selection criterion*. Observe that while equations 1 and 2 derive from (3.8) and from the definition of equivalent net representations, equation 3 does not. The latter equation is a compatibility request.

Our next aim is to prove that this criterion is indeed a generalization of (3.7). Consider a causal representation of the form  $\{\pi, \mathbb{1}\}$  satisfying the above selection criterion. Given  $o$  and  $N$  as above, since  $N$  is pathwise connected then  $W_a^{No}$  is constant, i.e. independent of  $a$  (see the first observation at the end of Section 2). So we can rewrite it as  $W^{No}$ . By the third equation of the selection criterion we have

$$W^{N'o} = W^{No} = W^{\tilde{N}o}, \quad N \subseteq N' \cap \tilde{N}.$$

This observation and, once again, pathwise connectedness of  $M$  implies that  $W^{No}$  is independent of the region  $N$ . So we have obtained the DHR notion of sharp excitation for topologically trivial representations.

Denote the set of representations satisfying the selection criterion by  $SC(\mathcal{A}_{K(M)})$ , and consider the  $C^*$ -subcategory of  $\text{Rep}^{net}(\mathcal{A}_{K(M)})$  whose set of objects is  $SC(\mathcal{A}_{K(M)})$ . We denote this category by the same symbol  $SC(\mathcal{A}_{K(M)})$  used to denote the corresponding set of objects. We denote the full  $C^*$ -subcategory of  $SC(\mathcal{A}_{K(M)})$  whose objects are topologically trivial net representations by  $SC_t(\mathcal{A}_{K(M)})$ . Because of the Borchers property, a routine calculation shows that these two categories are closed under direct sums and subobjects. Unitary equivalence classes of irreducible objects of  $SC(\mathcal{A}_{K(M)})$  are the *superselection sectors* of the theory, and the analysis of their charge structure and topological content will be our scope from now on. Note that the superselection sectors of the subcategory  $SC_t(\mathcal{A}_{K(M)})$  are the DHR-sectors analyzed in [53].

## 4 Net cohomology and the localization of the fundamental group

The category of the net representations satisfying the selection criterion admits an equivalent description in terms of the net cohomology of the poset  $K(M)$  with values in the

observable net. In the present section we prove a property of net cohomology which is at the base of this equivalence, the localization of the fundamental group, i.e., the representation of the first homotopy group defined by a 1-cocycle is localized. As we shall see in the following, this property is the key for understanding both charge structure and topological content of superselection sectors.

Consider the observable net in the reference representation  $\mathcal{R}_{K(M)}$ , and denote the category of 1-cocycles of the poset  $K(M)$  with values in the net  $\mathcal{R}_{K(M)}$  by  $Z^1(\mathcal{R}_{K(M)})$ . This is the  $C^*$ -subcategory of  $Z^1(K(M), \mathfrak{B}(\mathcal{H}_0))$  whose objects  $z$  and whose arrows  $t \in (z, \hat{z})$  satisfy the *locality condition*, i.e.,

$$z(b) \in \mathcal{R}(|b|) , \quad b \in \Sigma_1(K(M)) , \quad (4.1)$$

and

$$t_a \in \mathcal{R}(a) , \quad a \in \Sigma_0(K(M)) . \quad (4.2)$$

Now, as any 1-cocycle defines a representation of the fundamental group of the poset  $K(M)$ , we denote the full  $C^*$ -subcategory of  $Z^1(\mathcal{R}_{K(M)})$  whose objects are trivial representations of the fundamental group by  $Z_t^1(\mathcal{R}_{K(M)})$ . Sometimes we shall refer to the elements of  $Z_t^1(\mathcal{R}_{K(M)})$  as topologically trivial 1-cocycles. Note that topologically trivial 1-cocycles are nothing but that 1-coboundaries in  $Z^1(K(M), \mathfrak{B}(\mathcal{H}_0))$ .

Before diving further into the deep sea of net cohomology, some words of explanation are in order. As pointed out in Section 3.1, the set of indices of the observable net in a globally hyperbolic spacetime is non-directed under inclusion when the spacetime either is multiply connected or has compact Cauchy surfaces. In such situations it is not possible to define the  $C^*$ -algebra of all local observables, i.e., the  $C^*$ -inductive limit. This does not happen in Minkowski spacetime where there is a canonical choice of the set of indices, the set of double cones, which is directed under inclusion. This fact reflects in the way how DHR-sectors have been analyzed in these two situations. A key step of DHR analysis was the understanding that all the information of sectors is encoded in a unique Hilbert space: that one related to the vacuum. The category of DHR-sectors in Minkowski spacetime turns out to be equivalent to the category of localized and transportable endomorphisms of the *algebra of all local observables* defined in the vacuum representation [18]. In globally hyperbolic spacetimes DHR-sectors are still encoded in the vacuum Hilbert space (the reference) but in a different form. There is a notion of localized and transportable endomorphisms of the *observable net* defined in the vacuum representation, but the corresponding category is not equivalent to the category of DHR-sectors anymore when the set of indices is non-directed under inclusion [26]. The functor from the latter to the former category is not full, and it is not clear how to define the functor in the opposite direction. However, as pointed out by Roberts, in Minkowski spacetime DHR-sectors can be equivalently described in terms of the operators that in DHR analysis play the rôle of charge transporters: 1-cocycles of double cones taking values in the observable net defined in the vacuum representation [45, 47]. As shown in [26], this equivalence maintains in arbitrary globally hyperbolic spacetimes: the category  $SC_t(\mathcal{A}_{K(M)})$  is equivalent to the category  $Z_t^1(\mathcal{R}_{K(M)})$  (see also [53]). Analyzing the

category  $Z_t^1(\mathcal{R}_{K(M)})$ , the charge structure and the general covariance of DHR-sectors have been understood [26, 48, 53, 12].

Our first step for understanding sectors introduced in the present paper according to the selection criterion (3.8) consists in proving that the category  $\text{SC}(\mathcal{A}_{K(M)})$  is equivalent to  $Z^1(\mathcal{R}_{K(M)})$ . However, in this case, we have to pay attention, since non trivial topological objects are involved. To begin with, we show the property of net cohomology which underlies this equivalence.

Let  $\pi_1(K(M), a)$  be the first homotopy group of the poset  $K(M)$  based on the 0-simplex  $a$ . Given a 1-cocycle  $z$  of  $Z^1(\mathcal{R}_{K(M)})$ , we call the von Neumann algebra defined by

$$\mathcal{R}^z(M, a) \doteq \{z(\ell) : \ell \in \text{Loops}_{K(M)}(a)\}'' , \quad (4.3)$$

the *group algebra* associated with  $z$ , where the double prime stands for the bicommutant. The next theorem, to which we shall refer as the *localization of the fundamental group*, asserts that this von Neumann algebra is localized.

**Theorem 4.1.** *Given  $z \in Z^1(\mathcal{R}_{K(M)})$ , the following assertions hold :*

- (i)  $z(\ell) \in \mathcal{R}(a)$ ,  $\forall \ell \in \text{Loops}_{K(M)}(a)$ ;
- (ii)  $\mathcal{R}^z(M, a) \subseteq \mathcal{R}(a)$  .

*Proof.* Note that if we prove that  $z(\ell) \in \mathcal{R}(o)'$  for any diamond  $o$  such that  $o \perp a$ , then Haag duality implies that  $z(\ell) \in \mathcal{R}(a)$ . To this end, observe that if  $o \perp a$  then, by Corollary B.7, the loop  $\ell$  is homotopic to a loop  $\ell_1$  whose support<sup>2</sup> is contained in the causal complement of  $o$ . By homotopic invariance of 1-cocycles we have  $z(\ell) = z(\ell_1) \in \mathcal{R}(o)'$  completing the proof.  $\square$

In [26] it was proved that any 1-cocycle  $z$  of  $Z_t^1(\mathcal{R}_{K(M)})$  satisfies the following localization properties: *First*, given a path  $p$ , then

$$z(p) \in \mathcal{R}(o)', \quad o \perp \partial p, \quad (4.4)$$

where  $\partial p$  denotes the boundary of the path, i.e.,  $\partial p = \{\partial_0 p, \partial_1 p\}$ ; *secondly*, let  $p, q$  be paths with  $\partial_0 p = \partial_0 q$ , and let  $o$  be a diamond such that  $\partial_1 p, \partial_1 q \perp o$ , then

$$z(p) A z(p)^* = z(q) A z(q)^*, \quad A \in \mathcal{R}(o) . \quad (4.5)$$

Thanks to the localization of the fundamental group we now are able to prove that these properties hold in full generality.

**Corollary 4.2.** *Any 1-cocycle of  $Z^1(\mathcal{R}_{K(M)})$  satisfies (4.4) and (4.5).*

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<sup>2</sup>The support of a path is the union of the supports of the 1-simplices that form the path.



*Proof.* Let us prove (4.4). As the causal complement of  $o$  is pathwise connected, there is a path  $q$  with  $\partial q = \partial p$  and  $|q| \perp o$ . Observe that  $\bar{q} * p$  is a loop whose endpoint, say  $a$ , is causally disjoint from  $o$ . Since  $z(\bar{q} * p) \in \mathcal{R}(a)$  because of Theorem 4.1, we have  $z(p)A = z(q)z(\bar{q} * p)A = z(q)Az(\bar{q} * p) = Az(q)z(\bar{q} * p) = Az(p)$ , for any  $A \in \mathcal{R}(o)$  and this proves relation (4.4).

Let  $p, q$  and  $o$  be as in (4.5). As  $\bar{q} * p$  satisfies, with respect to  $o$ , the hypotheses of (4.4), we have  $z(p)Az(p)^* = z(q)z(\bar{q} * p)Az(\bar{q} * p)^*z(q)^* = z(q)Az(q)^*$ , for any  $A \in \mathcal{R}(o)$ .  $\square$

Now, the first application of this corollary is the crucial equivalence between sharply localized net representations and net cohomology.

**Theorem 4.3.**  $\text{SC}(\mathcal{A}_{K(M)})$  and  $Z^1(\mathcal{R}_{K(M)})$  are equivalent categories.

We prefer to postpone the, rather technical, proof of this equivalence in Appendix. We only point out that the functors that define the equivalence are an extension of the functors that define the equivalence between  $\text{SC}_t(\mathcal{A}_{K(M)})$  and  $Z_t^1(\mathcal{R}_{K(M)})$ . On these grounds the *superselection sectors* are described by the unitary equivalence classes of the irreducible objects of the category  $Z^1(\mathcal{R}_{K(M)})$ . So from now on, the analysis of superselection sectors will be carried out on  $Z^1(\mathcal{R}_{K(M)})$ .

Corollary 4.2 applies also to the analysis of the charge structure of superselection sectors which is the subject of the next section.

## 5 Charge Structure

The purpose of the present section is to show that the superselection sectors previously introduced manifest a charge structure. As observed in the previous section, this analysis will be performed on the category  $Z^1(\mathcal{R}_{K(M)})$ . At this point it is worth recalling that the charge structure of topologically trivial cocycles has been completely understood: the  $C^*$ -category  $Z_t^1(\mathcal{R}_{K(M)})$  has a tensor product, a permutation symmetry and a conjugation. This amounts to saying that the quantum numbers, i.e., the labels of sectors, have a composition law, a particle-antiparticle symmetry and an additional number saying that a sector has either the para-Bose or the para-Fermi statistics.

The study of the category  $Z_t^1(\mathcal{R}_{K(M)})$  resembles a standard argument of differential geometry. One first restricts the attention to the causal punctures of the spacetimes, namely to the categories  $Z_t^1(\mathcal{R}_{K(x^\perp)})$ . The advantage is that in these regions the point  $x$  has properties similar to the spacelike infinite in the Minkowski space. So one can prove the existence of a tensor product, permutation symmetry, left inverses and conjugated object in  $Z_t^1(\mathcal{R}_{K(x^\perp)})$  for any point  $x$ . Then, one observes that for 1-cocycle  $z$  of  $Z_t^1(\mathcal{R}_{K(M)})$  the local definitions, i.e. on the causal punctures, of tensor product, permutation symmetry and conjugated objects can be glued together to form the corresponding global notions.

It is now important to note that most of the constructions made for  $Z_t^1(\mathcal{R}_{K(M)})$ , the tensor product, the permutation symmetry and the conjugation do not involve the

topological triviality of 1-cocycles directly but rather relations (4.4) and (4.5). Thus, by Corollary 4.2, these constructions can be straightforwardly applied to  $Z^1(\mathcal{R}_{K(M)})$ . Only one point of that analysis cannot be extended to the general case: the proof of the existence of left inverses (and consequently the definition of statistics) because it relies on the fullness of the restriction functor from  $Z_t^1(\mathcal{R}_{K(M)})$  to  $Z_t^1(\mathcal{R}_{K(x^\perp)})$ , a property that does not hold for general 1-cocycles. However, although we will not prove the existence of left inverses for all the objects of  $Z^1(\mathcal{R}_{K(M)})$  we will be able to define objects with finite statistics via a detour.

We assume that the reader is familiar with symmetric tensor  $C^*$ -categories and related notions. Two references for this topic are [19, 38]. Other references whose focus is on the theory of superselection sectors are [47, 3, 48, 32].

## 5.1 DHR-like endomorphisms

Any 1-cocycle defines a class of endomorphisms that are localized and transportable in the same sense of those used in DHR analysis, but live on a presheaf associated with the observable net. Although these endomorphisms do not contain all the information about superselection sectors (see Remark 5.2), they enter the definitions of tensor product, permutation symmetry and conjugation.

Given a diamond  $o$ , the algebra of its *causal complement* is the  $C^*$ -algebra  $\mathcal{R}^\perp(o)$  generated by all the algebras  $\mathcal{R}(a)$  with  $a \perp o$ . The *presheaf*  $\mathcal{R}_{K(M)}^\perp$  associated with the observable net is the correspondence  $o \rightarrow \mathcal{R}^\perp(o)$ .

Consider a 1-cocycle  $z$  of  $Z^1(\mathcal{R}_{K(M)})$ . Fix a 0-simplex  $o$ , and let  $a$  be a 0-simplex such that  $a \perp o$ . Define

$$y_a^z(o)(A) \doteq z(p) A z(p)^*, \quad A \in \mathcal{R}^\perp(a), \quad (5.1)$$

where  $p$  is path with  $\partial_1 p \subseteq a$  and  $\partial_0 p = o$ . By (4.4) and (4.5) this definition does not depend on the path chosen  $p$  and on the choice of the starting point  $\partial_1 p$ . Therefore

$$y_a^z(o) \upharpoonright \mathcal{R}^\perp(a) = y_a^z(o), \quad \tilde{a} \subseteq a. \quad (5.2)$$

Fix a point  $x$  of the spacetime  $M$ . Since  $K$  is a base for the topology of  $M$ , the collection of 0-simplices  $K(x) \doteq \{\tilde{a} : x \in \tilde{a}\}$  is downward directed. The *stalk* in a point  $x$  can be seen either as the  $C^*$ -inductive limit of the system  $\mathcal{R}^\perp(o)$  with  $o \in K(x)$  or as the  $C^*$ -algebra generated by the algebras  $\mathcal{R}(o)$  for any  $o$  in  $K(x^\perp)$ .

Then, by property (5.2), the collection

$$y_x^z(o) \doteq \{y_a^z(o) \mid a \in K(x)\}, \quad o \in K(x^\perp), \quad (5.3)$$

is extendible to a morphism of the stalk  $\mathcal{R}^\perp(x)$ .

**Lemma 5.1.** *On the premises outlined before, we have that  $y_x^z(o)$  is an endomorphism of  $\mathcal{R}^\perp(x)$  satisfying the following properties:*

- (i)  $y_x^z(o) \upharpoonright \mathcal{R}(\tilde{o}) = id_{\mathcal{R}(\tilde{o})}$  for any  $\tilde{o} \in K(x^\perp)$  with  $\tilde{o} \perp o$ ;

- (ii)  $z(p) y_x^z(\partial_1 p) = y_x^z(\partial_0 p) z(p)$  for any path  $p$  in  $K(x^\perp)$ ;
- (iii)  $t_o y^z(o) = y^{z_1}(o) t_o$ , with  $t \in (z, z_1)$ ;
- (iv)  $y^z(o)(\mathcal{R}(\tilde{o})) \subseteq \mathcal{R}(\tilde{o})$  for any  $\tilde{o} \in K(x^\perp)$  with  $o \subseteq \tilde{o}$ .

The proof of these properties is the same as the proof of [53, Lemma 4.5]. We only observe that the first three properties are a consequence of the localization of the fundamental group (Properties (4.4) and (4.5)). Property (iv) derives from property (i) and from punctured Haag duality, because the restriction of  $\mathcal{R}_{K(M)}$  to  $K(x^\perp)$  satisfies Haag duality (see Section 3.2).

**Remark 5.2.** The family  $\{y_x^z(o) \mid o \in K(x^\perp)\}$  of endomorphisms of  $\mathcal{R}^\perp(x)$  are *localized* (i) and *transportable* (ii) in the same sense of DHR analysis. So, using the interpretation given in DHR analysis we may think of  $y_x^z(o)$  as a charge localized within the diamond  $o$  and the 1-cocycle  $z$  as the transporter of these charges. As said at the beginning of this section, these endomorphisms do not contain all the information of superselection sectors. Among the various difficulties, the simplest one is to observe that if  $z$  is an irreducible object but carries a non-trivial representation of the fundamental group then, by property (ii) of Lemma 5.1, the corresponding endomorphism is not irreducible.

We finally point out two useful relations. The first one, an obvious consequence of (5.2), says that

$$y_x^z(o) \upharpoonright \mathcal{R}^\perp(a) = y_{\tilde{x}}^z(o) \upharpoonright \mathcal{R}^\perp(a) , \quad (5.4)$$

where  $x, \tilde{x} \in a$  and  $o \in K(x^\perp) \cap K(\tilde{x}^\perp)$ . This, in turn, implies that

$$y_x^z(o)(z(p)) = y_{\tilde{x}}^z(o)(z(p)) , \quad (5.5)$$

for any pair  $x, \tilde{x}$  of points and any path  $p$  such that  $|p| \subseteq K(x^\perp) \cap K(\tilde{x}^\perp)$ . The proof of these two properties is given in [53] where they are called gluing conditions, because they allow to extend cocycles and arrows defined on causal punctures over all  $K(M)$ .

## 5.2 Tensor structure

Thanks to the localization of the fundamental group, we shall define the tensor product and the permutation symmetry in  $Z^1(\mathcal{R}_{K(M)})$  by the same formulas as those used to define the corresponding notions in  $Z_t^1(\mathcal{R}_{K(M)})$  [53]. In that paper these formulas are first defined on the causal punctures and after extended globally by the gluing conditions. For brevity we shall give directly the global definitions. Clearly, most of the proofs are omitted, with some exceptions because they need modifications from the original ones.

We start by introducing a preliminary definition. Given  $z, z_1 \in Z^1(\mathcal{R}_{K(M)})$  and  $t \in (z, z_1), s \in (z_2, z_3)$ , define

$$\begin{aligned} z(p) \times_x z_1(q) &\doteq z(p) y_x^z(\partial_1 p)(z_1(q)) , \quad p, q \text{ paths in } K(x^\perp) , \\ t_a \times_x s_{\tilde{a}} &\doteq t_a y_x^z(a)(s_{\tilde{a}}) , \quad a \in \Sigma_0(K(x^\perp)) . \end{aligned} \quad (5.6)$$

As a consequence of properties (ii) and (iii) of localized transportable endomorphisms (see Lemma 5.1) we have

$$z(p * \hat{p}) \times_x z_1(q * \hat{q}) = z(p) \times_x z_1(q) \ z(\hat{p}) \times_x z_1(\hat{q}) , \quad (5.7)$$

and

$$t_{\partial_0 p} \times_x s_{\partial_0 q} \ z(p) \times_x z_1(q) = z_2(p) \times_x z_3(q) \ t_{\partial_1 p} \times_x s_{\partial_1 q} , \quad (5.8)$$

(cfr. [53, Lemma 4.6]). Furthermore we have the following

**Lemma 5.3.** *Given a pair of paths  $p, q$  of  $K(x^\perp)$ . Then*

$$z(p) \times_x z_1(q) = z_1(q) \times_x z(p) ,$$

whenever  $\partial_i p \perp \partial_i q$  for  $i = 0, 1$ .

*Proof.* There are in  $K(x^\perp)$  two paths  $p_1 = b_{j_n} * \dots * b_{j_1}$  and  $q_1 = b_{k_n} * \dots * b_{k_1}$  such that  $|b_{j_i}| \perp |b_{k_i}|$  for  $i = 1, \dots, n$  and  $\partial p_1 = \partial p$ ,  $\partial q_1 = \partial q$ , see [53, Section 3.2.1]. For these paths we have  $z(p_1) \times_x z_1(q_1) = z_1(q_1) \times_x z_1(p_1)$  (cfr. [53, Lemma 4.8]). Using this and (5.7) we have

$$\begin{aligned} z(p) \times_x z_1(q) &= z(p_1) \times_x z_1(q_1) \ z(\overline{p_1} * p) \times_x z_1(\overline{q_1} * q) \\ &= z_1(q_1) \times_x z(p_1) \ z_1(\overline{q_1} * q) \times_x z(\overline{p_1} * p) \\ &= z_1(q) \times_x z(p) , \end{aligned}$$

where we have used the fact that

$$z(\overline{p_1} * p) \times_x z_1(\overline{q_1} * q) = z(\overline{p_1} * p) \ z_1(\overline{q_1} * q) = z_1(\overline{q_1} * q) \ z(\overline{p_1} * p) = z_1(\overline{q_1} * q) \times_x z(\overline{p_1} * p) ,$$

because  $z(\overline{p_1} * p) \in \mathcal{R}(\partial_1 p)$ ,  $z_1(\overline{q_1} * q) \in \mathcal{R}(\partial_1 q)$ ,  $\partial_1 p \perp \partial_1 q$ , and property (i) of localized and transportable endomorphisms of stalks.  $\square$

The tensor product is a particular case of the expressions (5.6). Given  $z, z_1 \in Z^1(\mathcal{R}_{K(M)})$  and  $t, s$  arrows of  $Z^1(\mathcal{R}_{K(M)})$  define

$$\begin{aligned} (z \otimes z_1)(b) &\doteq z(b) \times_x z_1(b) , \quad b \in \Sigma_1(K(M)) , \\ (t \otimes s)_a &\doteq t_a \times_{\tilde{x}} s_a , \quad a \in \Sigma_0(K(M)) , \end{aligned} \quad (5.9)$$

where  $x$  and  $\tilde{x}$  are points of  $M$  such that  $x \perp cl(|b|)$  and  $\tilde{x} \perp cl(a)$ . One first observes that these definitions behave as a tensor product when restricted to the causal puncture  $K(x^\perp)$  of  $M$  in  $x$ . Afterwards one observes that by (5.5) the definitions are independent of the choice of the point ([53, Proposition 4.7 and Lemma 4.17]).

The following lemma characterizes the morphism of stalks associated with the tensor product of two 1-cocycles.

**Lemma 5.4.** *Let  $z, \tilde{z} \in Z^1(\mathcal{R}_{K(M)})$ . Then  $y_x^{z \otimes z_1}(o) = y_x^z(o) y_x^{\tilde{z}_1}(o)$ , for any  $o \in K(x^\perp)$ .*

*Proof.* Let  $a$  be a 0-simplex in  $K(x)$  (see Section 5.2) such that  $cl(a) \perp o$ . Let  $p$  be a path in  $K(x^\perp)$  such that  $cl(\partial_1 p) \subset a$  and  $\partial_0 p = o$ . Take  $A \in \mathcal{R}^\perp(a)$ . According to the definition (5.3) we have

$$\begin{aligned} y_x^{z \otimes z_1}(o)(A) &= y_a^{z \otimes z_1}(o)(A) \\ &= (z \otimes z_1)(p) A (z \otimes z_1)(p)^* \\ &= (z(p) \times_x z_1(p)) A (z(p) \times_x z_1(p))^* \\ &= (z(p) y_x^z(\partial_1 p)(z_1(p))) A (z(p) y_x^z(\partial_1 p)(z_1(p)))^* \\ &= z(p) z(q) z_1(p) z(q)^* A (z(p) z(q) z_1(p) z(q)^*)^*, \end{aligned}$$

where  $q$  is a path in  $K(x^\perp)$  such that  $\partial_0 q = \partial_1 p$  and  $cl(\partial_1 q) \subset a$  and  $\partial_1 q \perp \partial_1 p$ .<sup>3</sup> Applying (4.4) we have  $z(q)^* A z(q) = A$ ; thus

$$\begin{aligned} y_x^{z \otimes z_1}(o)(A) &= z(p) z(q) z_1(p) A (z(p) z(q) z_1(p))^* \\ &= z(p * q) z_1(p) A (z(p * q) z_1(p))^* \\ &= z(p * q) y_x^{z_1}(o)(A) z(p * q)^* \\ &= y_x^z(o)(y_x^{z_1}(o)(A)) , \end{aligned}$$

where we have used the relation  $y_x^{z_1}(o)(A) \in \mathcal{R}^\perp(a)$  which is a consequence of property (iv) of localized and transportable endomorphisms of stalks.  $\square$

The permutation symmetry  $\varepsilon$  is defined, for any pair  $z, z_1$  in  $Z^1(\mathcal{R}_{K(M)})$ , by

$$\varepsilon(z, z_1)_a \doteq z_1(q)^* \times_x z(p)^* z(p) \times_x z_1(q) , \quad a \in \Sigma_0(K(M)) , \quad (5.10)$$

where  $x$  is any point of  $M$  with  $x \perp cl(a)$ , and  $p, q$  are two paths of  $K(x^\perp)$  with  $\partial_0 p \perp \partial_0 q$  and  $\partial_1 p = \partial_1 q = a$ . Once again one restricts the attention to the causal punctures. First one observes that the above definition does not depend on the choice of the paths  $p$  and  $q$ , and shows  $\varepsilon$  is indeed a symmetry in restriction to the causal punctures. Afterwards, one checks that the above definition does not depend on the choice of the point  $x$  (see [53]). A useful relation for analyzing the topological content of 1-cocycles is provided in the following lemma.

**Lemma 5.5.** *Given  $o$  and  $x \in M$  with  $cl(o) \perp x$ , then*

$$\varepsilon(z, z_1)_o z(\ell) \times_x z_1(\ell') = z_1(\ell') \times_x z(\ell) \varepsilon(z, z_1)_o ,$$

where  $\ell, \ell' \in \text{Loops}_{K(x^\perp)}(o)$ .

---

<sup>3</sup>Note that such a path exists since  $cl(\partial_1 p) \subset a$ , by Lemma B.5, there are two diamonds  $o_1$  and  $o_2$  such that  $cl(o_2) \subset a$  and  $cl(o_1) \perp cl(\partial_1 p)$ . So we can take  $q$  to be the 1-simplex whose support is  $o_2$ , the 1-face is  $o_2$ , and the 0-face is  $\partial_1 p$ .

*Proof.* Consider a point  $x$  and two paths  $p$  and  $q$  as in the definition of  $\varepsilon$ . By (5.7) we have

$$\begin{aligned}
\varepsilon(z, z_1)_o z(\ell) \times_x z_1(\ell') &= z_1(q)^* \times_x z(p)^* z(p) \times_x z_1(q) z(\ell) \times_x z_1(\ell') \\
&= z_1(q)^* \times_x z(p)^* z(p * \ell) \times_x z_1(q * \ell') \\
&= z_1(\ell') z_1(q * \ell')^* \times_x z(\ell) z(p * \ell)^* z(p * \ell) \times_x z_1(q * \ell') \\
&= z_1(\ell') \times_x z(\ell) z_1(q * \ell')^* \times_x z(p * \ell)^* z(p * \ell) \times_x z_1(q * \ell') \\
&= z_1(\ell') \times_x z(\ell) \varepsilon(z, z_1)_o
\end{aligned}$$

because  $p * \ell$  and  $q * \ell'$  are paths satisfying the definition of  $\varepsilon$ .  $\square$

Note that if we take in this lemma  $\ell'$  as the trivial loop, i.e.,  $\ell' = \sigma_0 o$  then  $\varepsilon(z, z_1)_o z(\ell) = y_x^{z_1}(o)(z(\ell)) \varepsilon(z, z_1)_o$ . Since the unitaries  $z(\ell)$  generate the algebra  $\mathcal{R}^z(M, o)$  and since  $y_x^{z_1}(o)$  is normal on this algebra we have

$$A = \varepsilon(z_1, z)_o y_x^{z_1}(o)(A) \varepsilon(z, z_1)_o, \quad A \in \mathcal{R}^z(M, o), \quad (5.11)$$

with  $o \in K(x^\perp)$ .

### 5.3 Statistics and Conjugation

Our purpose now is to identify the objects of  $Z^1(\mathcal{R}_{K(M)})$  having conjugates. The first step will be to understand what are the objects with finite statistics. To reach this goal we shall not follow the traditional way, rather we shall identify a  $C^*$ -subcategory  $\tilde{Z}^1(\mathcal{R}_{K(M)})$  closed under tensor product, direct sums, subobjects and having left inverses, and containing all the simple objects of  $Z^1(\mathcal{R}_{K(M)})$ . Within this category we shall define the objects with finite statistics in a same way as in DHR analysis. Afterwards, we prove that any object with finite statistics has conjugates.

We recall that a left inverse  $\phi$  of an object  $z$  of a tensor  $C^*$ -category is a family of linear mappings  $\phi_{z_1, z_2} : (z \otimes z_1, z \otimes z_2) \rightarrow (z_1, z_2)$ , for pair any  $z_1, z_2$  of objects, satisfying the following relations: given  $X \in (z \otimes z_1, z \otimes z_2)$ , then

$$\begin{aligned}
\phi_{z_1 \otimes \tilde{z}, z_2 \otimes \tilde{z}}(X \otimes 1_{\tilde{z}}) &= \phi_{z_1, z_2}(X) \otimes 1_{\tilde{z}}, \\
\phi_{z', z''}(1_z \otimes S \cdot X \cdot 1_z \otimes R) &= S \cdot \phi_{z_1, z_2}(X) \cdot R, \quad S \in (z_2, z''), R \in (z_1, z').
\end{aligned}$$

A left inverse of  $z$  is said to be *positive* whenever, for any object  $\tilde{z}$ ,  $\phi_{\tilde{z}, \tilde{z}}$  sends positive elements of  $(z \otimes \tilde{z}, z \otimes \tilde{z})$  in to positive elements of  $(\tilde{z}, \tilde{z})$ ; *normalized* whenever  $\phi_{i, i}(1_z) = 1_i$ . A positive left inverse  $\phi$  of  $z$  is said to be *faithful* whenever, for any object  $\tilde{z}$ ,  $\phi_{\tilde{z}, \tilde{z}}(X) \neq 0$  for any positive and non-zero element  $X$  of  $(z \otimes \tilde{z}, z \otimes \tilde{z})$ .

From now on, by a left inverse we will always mean a positive and normalized left inverse.

An object  $u \in Z^1(\mathcal{R}_{K(M)})$  is said to be *simple* whenever

$$\varepsilon(u, u) = \chi(u) \cdot 1_{u \otimes u}, \quad \text{where } \chi(u) \in \{1, -1\}. \quad (5.12)$$

If  $u$  is simple, it turns out that  $y_x^u(o) : \mathcal{R}^\perp(x) \rightarrow \mathcal{R}^\perp(x)$  is an *automorphism* for any  $o \in \Sigma_0(K(x^\perp))$  (cfr. [53, Proposition 4.12, Theorem 4.22]). If we denote the inverse of  $y_x^u(o)$  by  $y_x^u(o)^{-1}$  it is easily seen that

$$\phi_{z_1, z_2}(t)_o \doteq y_x^u(o)^{-1}(t_o) , \quad t \in (u \otimes z_1, u \otimes z_2) , \quad (5.13)$$

where  $x$  is a point of  $M$  causally disjoint from the closure of  $o$ , is a faithful left inverse of  $u$ . So any simple object of  $Z^1(\mathcal{R}_{K(M)})$  has faithful left inverses.

Denote by  $\tilde{Z}^1(\mathcal{R}_{K(M)})$  the full  $C^*$ -subcategory of  $Z^1(\mathcal{R}_{K(M)})$  whose objects have faithful left inverses. By applying formulas (A.1) (A.2) and (A.3) in [51], it easily follows that this category is closed under tensor product, direct sum, subobjects and equivalence. Furthermore, this category is not trivial. In fact, as observed above, any simple object of  $Z^1(\mathcal{R}_{K(M)})$  belongs to this category.

Since the category  $\tilde{Z}^1(\mathcal{R}_{K(M)})$  has left inverses and since it is closed under tensor product, direct sums and subobjects, it is possible to apply the mathematical machinery of DHR analysis to define and classify the objects with finite statistics (see references quoted at the beginning of Section 5). An object  $z$  of  $\tilde{Z}^1(\mathcal{R}_{K(M)})$  has *finite statistics* if it admits a standard left inverse  $\phi$ , that is

$$\phi_{z,z}(\varepsilon(z,z))^2 = c \cdot 1_z , \quad c > 0 .$$

Let  $Z^1(\mathcal{R}_{K(M)})_f$  be the full  $C^*$ -subcategory of  $\tilde{Z}^1(\mathcal{R}_{K(M)})$  whose objects with finite statistics. Then,  $Z^1(\mathcal{R}_{K(M)})_f$  is closed under tensor product, direct sum and subobjects. Any object of this category is a finite direct sum of irreducible objects with finite statistics. Given an irreducible object  $z$  of  $Z^1(\mathcal{R}_{K(M)})_f$  and a left inverse  $\phi$ , then

$$\phi_{z,z}(\varepsilon(z,z)) = \lambda(z) \cdot 1_z ,$$

where  $\lambda(z)$  is an invariant of the equivalence class of  $z$ , called the *statistics parameter*, and it is the product of two invariants:

$$\lambda(z) = \kappa(z) \cdot d(z)^{-1} \quad \text{where } \kappa(z) \in \{1, -1\} , \quad d(z) \in \mathbb{N} .$$

The possible statistics of  $z$  are classified by the *statistical phase*  $\kappa(z)$  distinguishing para-Bose (1) and para-Fermi (-1) statistics and by the *statistical dimension*  $d(z)$  giving the order of the para-statistics. Ordinary Bose and Fermi statistics correspond to  $d(z) = 1$ .

In a symmetric tensor  $C^*$ -category an object  $z$  has *conjugates* if there exists an object  $\bar{z}$  and a pair of arrows  $r \in (\iota, \bar{z} \otimes z)$  and  $\bar{r} \in (\iota, z \otimes \bar{z})$  satisfying the *conjugate* equations

$$\bar{r}^* \otimes 1_z \cdot 1_{\bar{z}} \otimes r = 1_{\bar{z}} , \quad r^* \otimes 1_{\bar{z}} \cdot 1_z \otimes \bar{r} = 1_z . \quad (5.14)$$

It is a well known fact that if an object has conjugates, then it has a faithful left inverse and finite statistics. So any object of  $z$  having conjugates belongs to  $Z^1(\mathcal{R}_{K(M)})_f$ . We now show that any object of this category has conjugates. To this end it is enough to prove that simple objects have conjugates. So, consider a simple object  $u$ . Define

$$\bar{u}(b) \doteq y_x^u(\partial_0 b)^{-1}(u(b)^*) , \quad b \in \Sigma_1(K(M)) , \quad (5.15)$$

for some  $x$  with  $x \perp cl(|b|)$ . Again, one first observe that the definition is independent of the choice of the point  $x$ . Finally, one checks that  $(u \otimes \bar{u})(b) = \mathbb{1}$  and that  $(\bar{u} \otimes u)(b) = \mathbb{1}$ . So, if we take  $r = \bar{r} = \mathbb{1}$ , then  $r$  and  $\bar{r}$  satisfy the conjugate equations for  $u$  and  $\bar{u}$ , cfr.[53]. Then, the above observation leads to the following conclusion.

**Theorem 5.6.** *Any object of  $Z^1(\mathcal{R}_{K(M)})_{\mathbf{f}}$  has conjugates.*

## 6 The topological content

The twofold information contained in 1-cocycles, the charge and the topological content, can be splitted. We shall see that any 1-cocycle  $z$  can be written as a suitable composition (the join) of two 1-cocycles: the charge component  $\langle z \rangle$ , a topologically trivial 1-cocycle having the same charge quantum numbers as  $z$ ; the topological component  $\chi_z$ , a 1-cocycle that carries the same representation of the fundamental group of  $M$  as  $z$  but it does not take values in the observable net. This decomposition holds for any 1-cocycle of  $Z^1(\mathcal{R}_{K(M)})$ . When we specialize to the finite statistics case, we shall find a relation between the statistics and the topological content of 1-cocycles. This relation shall lead us to discover a new invariant: the topological dimension.

In order to decompose 1-cocycles into charge and topological component, we introduce the notion of path-frame which assigns to any 0-simplex a path-coordinate with respect to a fixed 0-simplex, the pole. To be precise we fix a 0-simplex  $o$ , *the pole*. For any 0-simplex  $a$ , we pick a path  $p_{(a,o)}$  from  $o$  to  $a$  such that  $p_{(o,o)}$  is homotopic to the trivial loop over  $o$ , i.e., the degenerate 0-simplex  $\sigma_0 o$ . We call the collection  $P_o \doteq \{p_{(a,o)} \mid a \in \Sigma_0(K(M))\}$  a *path-frame* with pole  $o$ . The *translation* of a path-frame  $P_o$  is the path-frame  $P_o * o_1$  whose elements, denoted by  $p_{(a,o_1)}$ , are of the form  $p_{(a,o)} * \overline{p_{(o_1,o)}}$ . Note that the translation  $P_o * o_1 * o$  can be identified with  $P_o$  since they have homotopic elements.

### 6.1 Splitting

We now show the splitting of a 1-cocycle into charge and topological component.

Fix a path-frame  $P_o$  with pole  $o$ . Given a 1-cocycle  $z$  of  $Z^1(\mathcal{R}_{K(M)})$  define

$$\langle z \rangle(b) \doteq z(p_{(\partial_0 b, o)} * \overline{p_{(\partial_1 b, o)}}), \quad b \in \Sigma_1(K(M)). \quad (6.1)$$

We call  $\langle z \rangle$  the *charge component* of  $z$ .

It is very easy to see that  $\langle z \rangle$  is a topologically trivial 1-cocycle, i.e.,  $\langle z \rangle \in Z_t^1(\mathcal{R}_{K(M)})$ . In fact, it follows straightforwardly from the definition that  $\langle z \rangle$  is a 1-coboundary of  $Z^1(K(M), \mathfrak{B}(\mathcal{H}_0))$ . Moreover, given a 1-simplex  $b$  for any 0-simplex  $a$  with  $|b| \perp a$  by (4.4) we have  $\langle z \rangle(b) \in \mathcal{R}(a)'$ . Thus  $\langle z \rangle(b) \in \mathcal{R}(|b|)$  by Haag duality.

We now show that definition (6.1) is independent, up to equivalence, of the choice of the path-frame and of the choice of the pole. Given another path-frame  $Q_o$  define  $s_a \doteq z(q_{(a,o)} * \overline{p_{(a,o)}})$  for any 0-simplex  $a$ . Then

$$s_{\partial_0 b} z(p_{(\partial_0 b, o)} * \overline{p_{(\partial_1 b, o)}}) = z(q_{(\partial_0 b, o)} * \overline{p_{(\partial_1 b, o)}}) = z(q_{(\partial_0 b, o)} * \overline{q_{(\partial_1 b, o)}}) s_{\partial_1 b},$$



for any 1-simplex  $b$ . Furthermore, since  $q_{(a,o)} * \overline{p_{(a,o)}}$  is a loop over  $a$ , then  $s_a \in \mathcal{R}(a)$  because of the localization of the fundamental group. Thus, different choices of path-frames with the same pole lead to equivalent charge components of  $z$ . Now, consider another pole  $o_1$  and the translation  $P_o * o_1$  of  $P_o$ . Since 1-cocycles are homotopic invariant we have

$$z(p_{(\partial_0 b, o)} * \overline{p_{(\partial_1 b, o)}}) = z(p_{(\partial_0 b, o_1)} * \overline{p_{(\partial_1 b, o_1)}}) , \quad (6.2)$$

because the paths  $p_{(\partial_0 b, o)} * \overline{p_{(\partial_1 b, o)}}$  and  $p_{(\partial_0 b, o)} * \overline{p_{(o, o_1)}} * p_{(o, o_1)} * \overline{p_{(\partial_1 b, o)}} = p_{(\partial_0 b, o_1)} * \overline{p_{(\partial_1 b, o_1)}}$  are homotopic. This completes the proof of our claim.

We now go deep inside the relation between  $z$  and its charge component. We fix a path-frame  $P_o$ . The first important observation is that the morphisms of stalks associated with  $z$  and  $\langle z \rangle$  are equal. According to (5.3) and (5.1), it is enough to see that given a path  $q$  and a 0-simplex  $a$ , with  $|q| \perp a$ , and  $A \in \mathcal{R}(a)$ , then

$$z(q) A z(q)^* = z(p_{(\partial_0 q, o)} * \overline{p_{(\partial_1 q, o)}}) A z(p_{(\partial_0 q, o)} * \overline{p_{(\partial_1 q, o)}})^* = \langle z \rangle(q) A \langle z \rangle(q)^* ,$$

where we have applied (4.5) since the paths  $q$  and  $p_{(\partial_0 q, o)} * \overline{p_{(\partial_1 q, o)}}$  satisfy the hypotheses of that relation.

**Lemma 6.1.** *The mapping  $\mathcal{P}_c : Z^1(\mathcal{R}_{K(M)}) \rightarrow Z_t^1(\mathcal{R}_{K(M)})$ , which sends an object  $z$  to its charged component  $\langle z \rangle$ , with respect to a fixed path-frame  $P_o$ , and acts as the identity on arrows  $t \rightarrow t$ , defines a faithful and symmetric, covariant  $*$ -functor.*

*Proof.* It is easily seen that  $\mathcal{P}_c$  a faithful and covariant  $*$ -functor. We only observe that if  $t \in (z, z_1)$ , then

$$t_{\partial_0 b} \langle z \rangle(b) = t_{\partial_0 b} z(p_{(\partial_0 b, o)} * \overline{p_{(\partial_1 b, o)}}) = z_1(p_{(\partial_0 b, o)} * \overline{p_{(\partial_1 b, o)}}) t_{\partial_1 b} = \langle z_1 \rangle(b) t_{\partial_1 b} ,$$

for any 1-simplex  $b$ . We now prove that  $\mathcal{P}_c$  preserves the tensor product. Given a 1-simplex  $b$ , pick  $x \in M$  such that  $|b| \in K(x^\perp)$ . Moreover, pick pole  $o_1$  in  $K(x^\perp)$ . Given  $z, z_1$ , and using (6.2) we have

$$\begin{aligned} \mathcal{P}_c(z \otimes z_1)(b) &= \langle z \otimes z_1 \rangle(b) \\ &= z \otimes z_1 (p_{(\partial_0 b, o)} * \overline{p_{(\partial_1 b, o)}}) \\ &= z \otimes z_1 (p_{(\partial_0 b, o_1)} * \overline{p_{(\partial_1 b, o_1)}}) , \end{aligned}$$

where  $p_{(a, o_1)}$  is the path associated with the translation  $P_o * o_1$ . Since  $|b|, o_1$  are in  $K(x^\perp)$ , there are paths  $q_0$  and  $q_2$  in  $K(x^\perp)$  which are homotopic respectively to  $p_{(\partial_0 b, o_1)}$  and  $p_{(\partial_1 b, o_1)}$  (Corollary B.7). The previous equation and the homotopic invariance of 1-cocycles lead to  $\mathcal{P}_c(z \otimes z_1)(b) = z \otimes z_1(q_0 * q_1)$ . Finally, by definition of the tensor product, by (5.7) and by applying again homotopic invariance of 1-cocycles we have

$$\begin{aligned} \mathcal{P}_c(z \otimes z_1)(b) &= z \otimes z_1(q_0 * q_1) = z(q_0 * q_1) \times_x z(q_0 * q_1) \\ &= z(p_{(\partial_0 b, o_1)} * \overline{p_{(\partial_1 b, o_1)}}) \times_x z(p_{(\partial_0 b, o_1)} * \overline{p_{(\partial_1 b, o_1)}}) \\ &= \langle z \rangle(b) \times_x \langle z_1 \rangle(b) \\ &= \langle z \rangle \otimes \langle z_1 \rangle(b) , \end{aligned}$$

for any 1-simplex  $b$ . Note that we have used the fact that  $z$  and  $\langle z \rangle$  define the same morphisms of stalks, as observed just before this lemma. Finally we prove that  $\varepsilon(z, z) = \varepsilon(\langle z \rangle, \langle z \rangle)$ . To this end, using the same notation in (5.10), recall that  $\varepsilon(z, z)_a$  does not depend on the choice of the paths  $p$  and  $q$  in (5.10). Now the paths  $p_{(\partial_0 p, o)} * \overline{p_{(a, o)}}$  and  $p_{(\partial_0 q, o)} * \overline{p_{(a, o)}}$  have the same endpoints as  $p$  and  $q$  respectively. However we cannot replace in the definition of  $\varepsilon(z, z)_a$  the paths  $p$  and  $q$  by  $p_{(\partial_0 p, o)} * \overline{p_{(a, o)}}$  and  $p_{(\partial_0 q, o)} * \overline{p_{(a, o)}}$  respectively, since the latter do not belong to  $K(x^\perp)$  in general. However, we note that by Corollary B.7 (see the observation below the corollary) there are two paths  $p_1$  and  $q_1$  lying in  $K(x^\perp)$  which are homotopic to  $p_{(\partial_0 p, o)} * \overline{p_{(a, o)}}$  and  $p_{(\partial_0 q, o)} * \overline{p_{(a, o)}}$  respectively. So we have

$$\varepsilon(z, z)_a = z(q)^* \times_x z(p)^* z(p) \times_x z(q) = z(q_1)^* \times_x z(p_1)^* z(p_1) \times_x z(q_1) .$$

Observing (5.6) and applying homotopic invariance of 1-cocycles

$$\begin{aligned} z(p_1) \times_x z(q_1) &= z(p_1) y_x^z(\partial_0 p_1)(z(q_1)) \\ &= z(p_{(\partial_0 p, o)} * \overline{p_{(a, o)}}) y_x^z(\partial_0 p_1)(z(p_{(\partial_0 q, o)} * \overline{p_{(a, o)}})) \\ &= \langle z \rangle(p) y_x^z(\partial_0 p)(\langle z \rangle(q)) \\ &= \langle z \rangle(p) y_x^{\langle z \rangle}(\partial_0 p)(\langle z \rangle(q)) \\ &= \langle z \rangle(p) \times_x \langle z \rangle(q) , \end{aligned}$$

where we have used the identities  $z(p_{(\partial_0 p, o)} * \overline{p_{(a, o)}}) = \langle z \rangle(p) z(p_{(\partial_0 q, o)} * \overline{p_{(a, o)}}) = \langle z \rangle(q)$ , which derive from (6.1), and that  $z$  and  $\langle z \rangle$  define the same endomorphisms of stalks.  $\square$

**Remark 6.2.** Two observations on  $\mathcal{P}_c$  are in order. First, it easily follows from the definition that  $\mathcal{P}_c$  is a projection, i.e.  $\mathcal{P}_c \mathcal{P}_c = \mathcal{P}_c$ . Secondly, the functor  $\mathcal{P}_c$  is not full in general. In fact, assume that  $z$  carries a non-trivial representation of the fundamental group. Take  $\ell$  a loop over  $o$ , define  $t_a \doteq z(p_{(a, o)} * \ell * \overline{p_{(a, o)}})$  for any  $a \in \Sigma_0(K(M))$ . By the localization of the fundamental group  $t_a \in \mathcal{R}(a)$  for any 0-simplex. Moreover,  $t_{\partial_0 b} \langle z \rangle(b) = z(p_{(\partial_0 b, o)} * \ell * \overline{p_{(\partial_0 b, o)}}) z(p_{(\partial_0 b, o)} * \overline{p_{(\partial_1 b, o)}}) = z(p_{(\partial_0 b, o)}) z(\ell) z(\overline{p_{(\partial_1 b, o)}}) = \langle z \rangle(b) t_{\partial_1 b}$ . So,  $t \in (\langle z \rangle, \langle z \rangle)$ , while  $t \notin (z, z)$  in general.

We now introduce a second 1-cocycle encoding the topological content of  $z$ . Fix a path-frame  $P_o$ . Define

$$\chi_z(b) \doteq z(\overline{p_{(\partial_0 b, o)}} * b * p_{(\partial_1 b, o)}) , \quad b \in \Sigma_1(K(M)) . \quad (6.3)$$

We call  $\chi_z$  the *topological component* of  $z$ .

The topological component  $\chi_z$  is a 1-cocycle of  $K(M)$  taking values in  $\mathcal{R}(o)$ . In fact, by the localization of the fundamental group,  $\chi_z$  takes values in  $\mathcal{R}(o)$ . Moreover

$$\begin{aligned} \chi_z(\partial_0 c) \chi_z(\partial_2 c) &= z(\overline{p_{(\partial_{00} c, o)}} * \partial_0 c * p_{(\partial_{10} c, o)}) z(\overline{p_{(\partial_{02} c, o)}} * \partial_2 c * p_{(\partial_{12} c, o)}) \\ &= z(\overline{p_{(\partial_{01} c, o)}}) z(\partial_0 c) z(p_{(\partial_{02} c, o)}) z(\overline{p_{(\partial_{02} c, o)}}) z(\partial_2 c) z(p_{(\partial_{11} c, o)}) \\ &= z(\overline{p_{(\partial_{01} c, o)}}) z(\partial_1 c) z(p_{(\partial_{11} c, o)}) \\ &= \chi_z(\partial_1 c) , \end{aligned}$$

for any 2-simplex  $c$ . Hence  $\chi_z$  is a 1-cocycle of the category  $Z^1(K(M), \mathcal{R}(o))$ . We now observe that  $z$  and  $\chi_z$  contains the same topological information, namely

$$\chi_z(\ell) = z(\ell) , \quad \ell \in \text{Loops}_{K(M)}(o) \quad (6.4)$$

In words  $z$  and  $\chi_z$  define the same representation of the first homotopy group. In fact assume that  $\ell$  is of the form  $\ell = b_n * \cdots * b_1$ . Then

$$\begin{aligned} \chi_z(\ell) &= z(\overline{p_{(\partial_0 b_n, o)}} * b_n * p_{(\partial_1 b_n, o)}) z(\overline{p_{(\partial_0 b_{n-1}, o)}} * b_{n-1} * p_{(\partial_1 b_{n-1}, o)}) \cdots \\ &\quad \cdots z(\overline{p_{(\partial_0 b_2, o)}} * b_2 * p_{(\partial_1 b_2, o)}) z(\overline{p_{(\partial_0 b_1, o)}} * b_1 * p_{(\partial_1 b_1, o)}) \\ &= z(\overline{p_{(\partial_0 b_n, o)}}) z(b_n) z(b_{n-1}) z(p_{(\partial_1 b_{n-1}, o)}) \cdots \\ &\quad \cdots z(\overline{p_{(\partial_0 b_2, o)}}) z(b_2) z(b_1) z(p_{(\partial_1 b_1, o)}) \\ &= z(\overline{\sigma_0 o}) z(\ell) z(\sigma_0 o) \\ &= z(\ell) . \end{aligned}$$

Note that equation (6.4) says that the representation of the fundamental group carried by the topological component of a 1-cocycle depends neither on the choice of the path-frame nor on the choice of the pole.

**Remark 6.3.** We point out the geometrical meaning of the topological component of a 1-cocycle  $z$  of  $Z^1(\mathcal{R}_{K(M)})$ . We recall that 1-cocycles of a poset taking values in a group are flat connections of the poset (see [49, 50]). So the topological component  $\chi_z$  is a *holonomy* of the flat connection  $z$ . We also note that the definition of  $\chi_z$  is the same as the definition of the reduced connection in the Ambrose-Singer theorem for posets [49, Theorem 4.28].

Further informations about the relation between  $z$  and its topological and charge components will be obtained by means of the following embedding theorem.

**Theorem 6.4.** *Given  $z \in Z^1(\mathcal{R}_{K(M)})$ , fix a path-frame  $P_o$  and, for any  $X \in \mathcal{R}^z(M, o)$ , define*

$$\varrho_a(X) \doteq z(p_{(a, o)}) X z(p_{(a, o)})^* , \quad a \in \Sigma_0(K) . \quad (6.5)$$

*Denote the family of mappings  $X \rightarrow \varrho_a(X)$ ,  $a \in \Sigma_0(K(M))$ , by  $\varrho$ . Then*

(i)  $\varrho : \mathcal{R}^z(M, o) \rightarrow (\langle z \rangle, \langle z \rangle)$  *is a injective  $*$ -morphism;*

(ii)  $\varrho : \mathcal{Z}(\mathcal{R}^z(M, o)) \rightarrow (z, z)$ ,

*where  $\mathcal{Z}(\mathcal{R}^z(M, o)) \doteq \mathcal{R}^z(M, o) \cap \mathcal{R}^z(M, o)'$ , the centre of  $\mathcal{R}^z(M, o)$ .*

*Proof.* Let us start by observing that  $\varrho_a(X) \in \mathcal{R}(a)$ . To this end, let us consider a loop  $\ell$  over  $o$ . Observe that  $\varrho_a(z(\ell)) = z(p_{(a, o)} * \ell * \overline{p_{(a, o)}})$ , and that  $p_{(a, o)} * \ell * \overline{p_{(a, o)}}$  is a loop over  $a$ . Then  $\varrho_a(z(\ell))$  belongs to the von Neumann algebra  $\mathcal{R}^z(M, a)$ . Since the unitaries  $z(\ell)$  generate  $\mathcal{R}^z(M, o)$  and since  $\varrho_a$  is, clearly, normal we have  $\varrho_a(X) \in \mathcal{R}^z(M, a)$ . By Theorem 4.1 we have  $\varrho_a(X) \in \mathcal{R}(a)$ .

(i) Given a 1-simplex  $b$  we have

$$\begin{aligned}\varrho_{\partial_0 b}(X) \langle z \rangle(b) &= z(p_{(\partial_0 b, o)}) X z(p_{(\partial_0 b, o)})^* z(p_{(\partial_0 b, o)}) z(p_{(\partial_1 b, o)})^* \\ &= z(p_{(\partial_0 b, o)}) X z(p_{(\partial_1 b, o)})^* \\ &= \langle z \rangle(b) \varrho_{\partial_0 b}(X) .\end{aligned}$$

Thus  $\varrho_{\partial_0 b}(X) \in (\langle z \rangle, \langle z \rangle)$ .

(ii) Assume that  $X$  belongs to the centre of  $\mathcal{R}^z(M, o)$ . Given a 1-simplex  $b$  we have

$$\begin{aligned}\varrho_{\partial_0 b}(X) z(b) &= z(p_{(\partial_0 b, o)}) X z(p_{(\partial_0 b, o)})^* z(b) \\ &= z(p_{(\partial_0 b, o)}) X z(\overline{p_{(\partial_0 b, o)}} * b * p_{(\partial_1 b, o)}) z(p_{(\partial_1 b, o)})^* \\ &= z(p_{(\partial_0 b, o)}) z(\overline{p_{(\partial_0 b, o)}} * b * p_{(\partial_1 b, o)}) X z(p_{(\partial_1 b, o)})^* \\ &= z(b) z(p_{(\partial_1 b, o)}) X z(p_{(\partial_1 b, o)})^* \\ &= z(b) \varrho_{\partial_0 b}(X) ,\end{aligned}$$

because  $\overline{p_{(\partial_0 b, o)}} * b * p_{(\partial_1 b, o)}$  is a loop over  $o$ .  $\square$

The first application of Theorem 6.4 derives from the following observation. Given a 1-cocycle  $z$ , define  $(z, z)_a \doteq \{t_a \mid t \in (z, z)\}$  as the component  $a$  of the set of intertwiners of the 1-cocycle. It is easily seen that any component of the algebra of intertwiners is a von Neumann algebra and that, since the poset is pathwise connected, this algebra is isomorphic to the full algebra of intertwiners. Now, since, by definition,  $p_{(o, o)}$  is homotopic to  $\sigma_0 o$  we have  $\varrho(X)_o = X$  for any  $X \in \mathcal{R}^z(M, o)$ . Then by Theorem 6.4 we have

$$\mathcal{R}^z(M, o) \subseteq (\langle z \rangle, \langle z \rangle)_o , \quad (6.6)$$

$$\mathcal{Z}(\mathcal{R}^z(M, o)) \subseteq (z, z)_o . \quad (6.7)$$

The next result is a direct consequence of (6.7).

**Corollary 6.5.** *Let  $z$  be a 1-cocycle of  $Z^1(\mathcal{R}_{K(M)})$ . If  $z$  is irreducible, then the representation of  $\pi_1(M, o)$  associated with  $z$  is a factor representation, i.e.,  $\mathcal{Z}(\mathcal{R}^z(M, o)) = \mathbb{C}\mathbb{1}$ .*

## 6.2 Joining

We learned that a 1-cocycle  $z$  splits in a pair  $(\chi_z, \langle z \rangle)$  of 1-cocycles:  $\langle z \rangle$  is topologically trivial but contains the charge structure of  $z$  (this will be clearly shown in Subsection 6.3);  $\chi_z$  encodes the topological content of the  $z$ . We now show that two such 1-cocycles can be joined together to form a 1-cocycle of  $Z^1(\mathcal{R}_{K(M)})$ .

Consider a topologically trivial 1-cocycle  $z \in Z_t^1(\mathcal{R}_{K(M)})$ , and a 1-cocycle  $\varphi$  of the category  $Z^1(K(M), \mathcal{R}(o))$ . We say that  $\varphi$  and  $z$  are *joinable* whenever  $\varphi(b) \in (z, z)_o$  for any 1-simplex  $b$ , and define the *join* of  $\varphi$  and  $z$ , with respect to a path-frame  $P_o$ , as

$$(\varphi \bowtie z)(b) \doteq z(b) z(p_{(\partial_1 b, o)}) \varphi(b) z(p_{(\partial_1 b, o)})^* , \quad b \in \Sigma_1(K(M)) . \quad (6.8)$$

Our aim is to show that the join  $\varphi \bowtie z$  is 1-cocycle of  $Z^1(\mathcal{R}_{K(M)})$ , whose topological and charge component are equivalent to  $\varphi$  and  $z$ , respectively, and that any 1-cocycle of  $Z^1(\mathcal{R}_{K(M)})$  arises as the join of its topological component and its charge component (note that, because of (6.6), these are joinable).

We start by showing a property of the join.

**Lemma 6.6.** *Fix a path-frame  $P_o$ . Let  $z \in Z_t^1(\mathcal{R}_{K(M)})$  and  $\varphi \in Z^1(K(M), \mathcal{R}(o))$  be joinable. Then*

$$(\varphi \bowtie z)(q) = z(p_{(\partial_0 q, o)}) \varphi(q) z(p_{(\partial_1 q, o)})^* ,$$

for any path  $q$ .

*Proof.* We start by observing that a 1-cocycle  $z$  is topologically trivial if, and only if, it is path-independent, that is,  $z(p) = z(\tilde{p})$  for any pair  $p, \tilde{p}$  of paths with the same endpoint. With this in mind, note that the above formula holds for paths which are formed by a single 1-simplex. By induction, assume that the formula holds for paths formed by  $n$  1-simplices. Let  $q$  be a path formed by  $n$  1-simplices and consider the path  $q_1 = q * b$ . Then

$$\begin{aligned} (\varphi \bowtie z)(q_1) &= (\varphi \bowtie z)(q) (\varphi \bowtie z)(b) \\ &= z(q) z(p_{(\partial_1 q, o)}) \varphi(q) z(p_{(\partial_1 q, o)})^* z(b) z(p_{(\partial_1 b, o)}) \varphi(b) z(p_{(\partial_1 b, o)})^* \\ &= z(q) z(p_{(\partial_1 q, o)}) \varphi(q) z(\overline{p_{(\partial_1 q, o)}} * b * p_{(\partial_1 b, o)}) \varphi(b) z(p_{(\partial_1 b, o)})^* \\ &= z(q) z(p_{(\partial_1 q, o)}) \varphi(q) \varphi(b) z(p_{(\partial_1 b, o)})^* \\ &= z(p_{(\partial_0 q, o)}) \varphi(q * b) z(p_{(\partial_1 b, o)})^* \\ &= z(p_{(\partial_0 q_1, o)}) \varphi(q_1) z(p_{(\partial_1 q_1, o)})^* , \end{aligned}$$

where we have used topological triviality of  $z$  ( $z$  gives the same value on paths having the same endpoints).  $\square$

Note that as a consequence of the above lemma, the join does not depend on the choice of the path-frame. This follows directly from the formula in the statement of Lemma 6.6 and from the topological triviality of the 1-cocycle  $z$ .

We now are in a position to prove the main result of this section.

**Theorem 6.7.** *Fix a path-frame  $P_o$ . Let  $z \in Z_t^1(\mathcal{R}_{K(M)})$  and  $\varphi \in Z^1(K(M), \mathcal{R}(o))$  be joinable. Then the following assertions hold.*

- (i) *The join  $\varphi \bowtie z$  is a 1-cocycle of  $Z^1(\mathcal{R}_{K(M)})$  whose topological and charge components are equivalent to  $\varphi$  and  $z$ , respectively.*
- (ii) *Any 1-cocycle  $z$  of  $Z^1(\mathcal{R}_{K(M)})$  is the join  $\chi_z \bowtie \langle z \rangle$ , with respect to  $P_o$ , of its topological component  $\chi_z$  with its charged component  $\langle z \rangle$ .*

*Proof.* (i) Clearly by Lemma 6.6 the join satisfies the 1-cocycle identity. Moreover  $\varphi \bowtie z$  is localized. In fact take a 1-simplex  $b$  which lies in  $K(x^\perp)$  and let  $a$  be a 0-simplex in  $K(x^\perp)$  such that  $a \perp |b|$ . Then

$$\begin{aligned}
(\varphi \bowtie z)(b) A &= z(b) z(p_{(\partial_1 b, o)}) \varphi(b) z(p_{(\partial_1 b, o)})^* A \\
&= z(b) z(p_{(\partial_1 b, o)}) \varphi(b) y_x^z(o)(A) z(p_{(\partial_1 b, o)})^* \\
&= z(b) z(p_{(\partial_1 b, o)}) y_x^z(o)(A) \varphi(b) z(p_{(\partial_1 b, o)})^* \\
&= z(b) y_x^z(\partial_1 b)(A) z(p_{(\partial_1 b, o)}) \varphi(b) z(p_{(\partial_1 b, o)})^* \\
&= z(b) y_x^z(\partial_1 b)(A) z(p_{(\partial_1 b, o)}) \varphi(b) z(p_{(\partial_1 b, o)})^* \\
&= y_x^z(\partial_0 b)(A) z(b) z(p_{(\partial_1 b, o)}) \varphi(b) z(p_{(\partial_1 b, o)})^* \\
&= A (\varphi \bowtie z)(b) ,
\end{aligned}$$

where we have used the properties of localized transportable endomorphisms of stalks and the fact that  $\chi \in (z, z)_o$ . By Haag duality  $(\varphi \bowtie z)(b) \in \mathcal{R}(|b|)$ ; hence  $\varphi \bowtie z \in Z^1(\mathcal{R}_{K(M)})$ . We now prove that the topological component  $\chi_{\varphi \bowtie z}$  is equivalent to  $\varphi$ . First of all, given a loop  $\ell = b_n * \dots * b_1$  over  $o$ , observe that

$$\begin{aligned}
\chi_{\varphi \bowtie z}(\ell) &= (\varphi \bowtie z)(\ell) \\
&= z(b_n) z(p_{(\partial_1 b_n, o)}) \varphi(b_n) z(\overline{p_{(\partial_1 b_n, o)}} * b_{n-1} * p_{(\partial_1 b_{n-1}, o)}) \varphi(b_{n-1}) \dots \\
&\quad \dots \varphi(b_2) z(\overline{p_{(\partial_1 b_2, o)}} * b_1 * p_{(\partial_1 b_1, o)}) \varphi(b_1) z(\overline{p_{(\partial_1 b_1, o)}}) \\
&= z(b_n * p_{(\partial_1 b_n, o)}) \varphi(b_n) \varphi(b_{n-1}) \dots \varphi(b_2) \varphi(b_1) z(p_{(\partial_1 b_1, o)}) \\
&= z(p_{(o, o)}) \varphi(\ell) z(p_{(o, o)}) \\
&= \varphi(\ell) ,
\end{aligned}$$

because  $z(\overline{p_{(\partial_1 b_{k+1}, o)}} * b_k * p_{(\partial_1 b_k, o)}) = \mathbb{1}$ , with  $k = 1, \dots, n-1$ , since  $z$  is topologically trivial. So define  $s_a \doteq \varphi(p_{(a, o)})$  for any 0-simplex  $a$ . Then  $s_a \in \mathcal{R}(a)$  and by the above identity we have

$$\begin{aligned}
s_{\partial_0 b} \chi_{\varphi \bowtie z}(b) &= \varphi(p_{(\partial_0 b, o)}) (\varphi \bowtie z)(\overline{p_{(\partial_0 b, o)}} * b * p_{(\partial_1 b, o)}) \\
&= \varphi(p_{(\partial_0 b, o)}) \varphi(\overline{p_{(\partial_0 b, o)}} * b * p_{(\partial_1 b, o)}) \\
&= \varphi(b) \varphi(p_{(\partial_1 b, o)}) \\
&= \varphi(b) s_{\partial_1 b} ,
\end{aligned}$$

for any 1-simplex  $b$ ; thus  $\chi_{\varphi \bowtie z}$  is equivalent to  $\varphi$  in  $Z^1(K(M), \mathcal{R}(o))$ . We prove that  $\langle \varphi \bowtie z \rangle$  is equivalent to  $z$  in  $Z_t^1(\mathcal{R}_{K(M)})$ . Define  $t_a \doteq z(p_{(a, o)}) \varphi(p_{(a, o)})^* z(p_{(a, o)})^*$ , for any 0-simplex  $a$ . We first observe that  $t_a \in \mathcal{R}(a)$ . In fact, since the topological component of the join is  $\varphi$  we have that  $\mathcal{R}^{\varphi \bowtie z}(o) = \mathcal{R}^\varphi(o)$ . Now, by Lemma 6.2 and using topological

triviality of  $z$  we have

$$\begin{aligned}
\varrho_a^{\varphi \bowtie z}(\varphi(p_{(a,o)})^*) &= (\varphi \bowtie z)(p_{(a,o)}) \varphi(p_{(a,o)})^* (\varphi \bowtie z)(\overline{p_{(a,o)}}) \\
&= z(p_{(a,o)}) \varphi(p_{(a,o)}) z(p_{(a,o)})^* \varphi(p_{(a,o)})^* z(p_{(a,o)}) \varphi(p_{(a,o)})^* z(p_{(a,o)})^* \\
&= z(p_{(a,o)}) \varphi(p_{(a,o)})^* z(p_{(a,o)})^* \\
&= t_a ,
\end{aligned}$$

for any 0-simplex  $a$ , where  $\varrho^{\varphi \bowtie z}$  is the embedding (6.5) associated with the join. The preceding observation and Theorem 6.4 imply that  $t_a \in \mathcal{R}(a)$ . According to (6.1) and by Lemma 6.6 we have

$$\begin{aligned}
t_{\partial_0 b} \langle \varphi \bowtie z \rangle(b) &= t_{\partial_0 b} (\varphi \bowtie z)(p_{(\partial_0 b, o)} * \overline{p_{(\partial_1 b, o)}}) \\
&= t_{\partial_0 b} z(p_{(\partial_0 b, o)}) \varphi(p_{(\partial_0 b, o)} * \overline{p_{(\partial_1 b, o)}}) z(p_{(\partial_1 b, o)})^* \\
&= z(p_{(\partial_0 b, o)}) \varphi(p_{(\partial_0 b, o)})^* \varphi(p_{(\partial_0 b, o)} * \overline{p_{(\partial_1 b, o)}}) z(p_{(\partial_1 b, o)})^* \\
&= z(p_{(\partial_0 b, o)}) \varphi(p_{(\partial_1 b, o)})^* z(p_{(\partial_1 b, o)})^* \\
&= z(p_{(\partial_0 b, o)}) z(p_{(\partial_1 b, o)})^* z(p_{(\partial_1 b, o)}) \varphi(p_{(\partial_1 b, o)})^* z(p_{(\partial_1 b, o)})^* \\
&= z(b) t_{\partial_1 b} ,
\end{aligned}$$

for any 1-simplex  $b$ , and this proves the equivalence.

(ii) As already observed if  $z \in Z^1(\mathcal{R}_{K(M)})$ , then  $\chi_z$  and  $\langle z \rangle$  are joinable. Then

$$\begin{aligned}
(\chi_z \bowtie \langle z \rangle)(b) &= \langle z \rangle(b) \langle z \rangle(p_{(\partial_1 b, o)}) \chi_z(b) \langle z \rangle(\overline{p_{(\partial_1 b, o)}}) \\
&= z(p_{(\partial_0 b, o)} * \overline{p_{(\partial_1 b, o)}}) z(p_{(\partial_1 b, o)}) z(\overline{p_{(\partial_0 b, o)}} * b * p_{(\partial_1 b, o)}) z(p_{(\partial_1 b, o)})^* \\
&= z(b) ,
\end{aligned}$$

for any 1-simplex  $b$ , and this completes the proof.  $\square$

As an easy consequence of this theorem, we have the following

**Corollary 6.8.** *Assume that the fundamental group of  $M$  is Abelian. Fix a path-frame  $P_o$ . Then for any irreducible 1-cocycle  $z$  we have that  $\chi_z \in Z^1(K(M), \mathbb{C})$  and*

$$z(b) = \chi_z(b) \langle z \rangle(b) , \quad b \in \Sigma_1(K(M)) .$$

*Proof.* Since  $\pi_1(M, o)$  is Abelian the algebra  $\mathcal{R}^z(M, o)$  is Abelian. Hence  $\mathcal{R}^z(M, o) = \mathbb{C}\mathbb{1}$  because of Corollary 6.5(ii). The proof now follows from Theorem 6.7.  $\square$

### 6.3 The topological dimension

We now focus on irreducible 1-cocycles having finite statistics. We shall see that the charge component completely encodes the charge structure of 1-cocycles and find a relation between the statistical dimension and the topological content of 1-cocycles: the representation of the fundamental group carried by the 1-cocycle is, up to infinite multiplicity, irreducible and finite dimensional. The dimension of this representation is bounded from above by the statistical dimension, and is a new invariant of sectors: the topological dimension.

**Proposition 6.9.** *Let  $z$  be an irreducible object of  $Z^1(\mathcal{R}_{K(M)})_{\text{f}}$  whose statistical parameter  $\lambda(z)$  is equal to  $\kappa(z)d(z)^{-1}$ . Fix a path-frame  $P_o$ . Then the following assertions hold.*

- (i) *If  $r$  and  $\bar{r}$  solve the conjugate equations for  $z$  and  $\bar{z}$ , then the same arrows solve the conjugate equations for  $\langle z \rangle$  and  $\langle \bar{z} \rangle$ .*
- (ii)  *$\langle z \rangle$  is an object with finite statistics which is, in general, a finite direct sum  $\langle z \rangle = z_1 \oplus \cdots \oplus z_m$ , with  $m \leq d(z)$ , of irreducible objects of  $Z_t^1(\mathcal{R}_{K(M)})_{\text{f}}$  having the same statistical phase as  $z$  and whose statistical dimension  $d(z_i)$  satisfies  $d(z) = d(z_1) + \cdots + d(z_m)$ .*

*Proof.* (i) The mapping  $\mathcal{P}_c : Z^1(\mathcal{R}_{K(M)}) \rightarrow Z_t^1(\mathcal{R}_{K(M)})$ , sending an object  $z$  into its charge component  $\langle z \rangle$ , with respect to  $P_o$ , and acting as the identity on arrows is a faithful and symmetric tensor  $*$ -functor (Lemma 6.1). It follows straightforwardly from these properties that if  $r$  and  $\bar{r}$  solve the conjugate equations for  $z$  and  $\bar{z}$ , then  $\mathcal{P}_c(r) = r$  and  $\mathcal{P}_c(\bar{r}) = \bar{r}$  solve the conjugate equations for  $\langle z \rangle$  and  $\langle \bar{z} \rangle$ . This, in particular, implies that  $\langle z \rangle$  has finite statistics.

(ii) Given  $z', z'' \in Z_t^1(\mathcal{R}_{K(M)})_{\text{f}}$ , define

$$\psi_{z', z''}(X) \doteq \frac{1}{d(z)} \mathcal{P}_c(r)^* \otimes 1_{z''} \cdot 1_{\mathcal{P}_c(\bar{z})} \otimes X \cdot \mathcal{P}_c(r) \otimes 1_{z'} , \quad (6.9)$$

with  $X \in (\mathcal{P}_c(z) \otimes z', \mathcal{P}_c(z) \otimes z'')$ <sup>4</sup>. By [38, Prop.4.5], the collection  $\psi \doteq \{\psi_{z', z''} \mid z', z'' \in Z_t^1(\mathcal{R}_{K(M)})_{\text{f}}\}$ , defines a standard left inverse of  $\langle z \rangle$ , within the category  $Z_t^1(\mathcal{R}_{K(M)})_{\text{f}}$ . Moreover  $\langle z \rangle$  has the same statistical dimension as  $z$ . In addition, since  $\mathcal{P}_c$  is symmetric we have

$$\begin{aligned} \psi_{\langle z \rangle, \langle z \rangle}(\varepsilon(\langle z \rangle, \langle z \rangle)) &= d(z)^{-1} \mathcal{P}_c(r)^* \otimes 1_{\langle z \rangle} \cdot 1_{\mathcal{P}_c(\bar{z})} \otimes \varepsilon(\langle z \rangle, \langle z \rangle) \cdot \mathcal{P}_c(r) \otimes 1_{\langle z \rangle} \\ &= d(z)^{-1} \mathcal{P}_c(r^* \otimes 1_z) \cdot \mathcal{P}_c(1_{\bar{z}} \otimes \varepsilon(z, z) \cdot \mathcal{P}_c(r \otimes 1_z)) \\ &= d(z)^{-1} \mathcal{P}_c(r^* \otimes 1_z \cdot 1_{\bar{z}} \otimes \varepsilon(z, z) \cdot r^* \otimes 1_z) \\ &= \mathcal{P}_c(\lambda(z) 1_z) \\ &= \lambda(z) 1_{\langle z \rangle} . \end{aligned}$$

It is a well known fact, see for instance [18], that this relation implies that  $\langle z \rangle$  is a finite direct sums of irreducible object of  $Z_t^1(\mathcal{R}_{K(M)})_{\text{f}}$  which have the same statistical phase as  $z$  and the sum of their statistical dimensions is equal to  $d(z)$ .  $\square$

**Proposition 6.10.** *The following assertions hold for any irreducible 1-cocycle  $z$  with finite statistics.*

- (i) *Let  $r, \bar{r}$  be arrows solving the conjugate equations for  $z$  and  $\bar{z}$ . Then the functional*

$$\omega_o^z(X) \doteq \frac{1}{d(z)} r_o^* y_x^{\bar{z}}(o)(X) r_o , \quad X \in \mathcal{R}^z(M, o) , \quad (6.10)$$

---

<sup>4</sup>Note that  $\psi_{z', z''}(X) = d(z)^{-1} r^* \otimes 1_{z''} \cdot 1_{\bar{z}} \otimes X \cdot r \otimes 1_{z'}$  with  $X \in (\langle z \rangle \otimes z', \langle z \rangle \otimes z'')$ , according to the definition of  $\mathcal{P}_c$ .



is a normal and faithful tracial state of  $\mathcal{R}^z(M, o)$ .

(ii) The algebra  $\mathcal{R}^z(M, o)$  is a type  $I_m$  factor with  $m \leq d(z)$ .

*Proof.* Fix a path-frame  $P_o$  and consider the charge component  $\langle z \rangle$ , with respect to  $P_o$ , of  $z$ .

(i) We have seen in the proof of Proposition 6.9 that  $\langle z \rangle$  has a standard left inverse  $\psi$  in  $Z_t^1(\mathcal{R}_{K(M)})$ , defined by equation (6.10). This implies that  $\psi_{\iota, \iota}$  is a faithful tracial state of the algebra  $(\langle z \rangle, \langle z \rangle)_o$ . Observe in particular that

$$\psi_{\iota, \iota}(t)_o = \frac{1}{d(z)} (r^* \otimes 1_\iota)_o (1_{\langle \bar{z} \rangle} \otimes t)_o (r \otimes 1_\iota)_o = \frac{1}{d(z)} r_o^* y_x^{\bar{z}}(o)(t_o) r_o = \omega_o^z(t_o) .$$

Hence  $\omega_o^z$  is a faithful tracial state of  $(\langle z \rangle, \langle z \rangle)_o$  because so is  $\psi_{\iota, \iota} \upharpoonright (\langle z \rangle, \langle z \rangle)_o$ . Now the proof follows by (6.6) and by observing that the morphisms of stalks of 1-cocycles are locally normal.

(ii) Note that  $(\langle z \rangle, \langle z \rangle)_o$  is a finite dimensional algebra having at most  $d(z)$  minimal mutually orthogonal projection. According to the definition of a finite type  $I$  factor (see [35]), the proof follows by Corollary 6.5 and by (6.6).  $\square$

On the ground of this result we can introduce the following notion.

**Definition 6.11.** *Given a 1-cocycle  $z$  with finite statistics, the **topological dimension**  $\tau(z)$  of  $z$  is the dimension of the factor  $\mathcal{R}^z(M, o)$ .*

Let us see which are the main properties of this new notion and its meaning. *First*, the topological dimension is a quantum number of superselection sectors, i.e., it is an invariant of the equivalence class of a 1-cocycle, since  $\mathcal{R}^z(M, o)$  is spatially equivalent to  $\mathcal{R}^{z_1}(M, o)$  whenever  $z$  is equivalent to  $z_1$ . *Secondly*, the topological dimension is bounded from above by the statistical dimension ((ii) of Proposition 6.10). *Thirdly*, the topological dimension and the tracial state defined in (6.10) characterize the topological content of a 1-cocycle. To explain this point we need a preliminary result which is an easy consequence of Proposition 6.10.

**Corollary 6.12.** *Let  $z$  be an irreducible object with finite statistics. Then:*

- (i)  $z$ , as a representation of  $\pi_1(M, o)$ , is equivalent to a representation of the form  $\mathbb{1}_{\mathcal{H}_0} \otimes \sigma_z$  where  $\sigma_z$  is a  $\tau(z)$ -dimensional irreducible representation of  $\pi_1(M, o)$ ;
- (ii) the normalized character  $c_{\sigma_z}$  of  $\sigma_z$  satisfies the equation

$$c_{\sigma_z}([\ell]) = \omega_o^z(z(\ell)) , \quad \ell \in \text{Loops}_{K(M)}(o) ,$$

where  $\omega_o^z$  is the functional (6.10)

*Proof.* (i) Since  $\mathcal{R}^z(M, o)$  is a type  $I_{\tau(z)}$  factor, there is a unitary operator  $U : \mathcal{H}_0 \rightarrow \mathcal{H}_0 \otimes C^{\tau(z)}$  such that  $\mathcal{R}^z(M, o) \cong \mathbb{1}_{\mathcal{H}_0} \otimes \mathbb{M}_{\tau(z)}$ . Then the representation  $\sigma_z$  of  $\pi_1(M, o)$  defined by

$$\mathbb{1}_{\mathcal{H}_0} \otimes \sigma_z([\ell]) \doteq U z(\ell) U^* \quad \ell \in \text{Loops}_{K(M)}(o) ,$$

is irreducible, because  $\mathbb{1}_{\mathcal{H}_o} \otimes (\sigma_z)'' = (\mathbb{1}_{\mathcal{H}_o} \otimes \sigma_z)'' = U \mathcal{R}^z(M, o) U^* = \mathbb{1} \otimes M_{\tau(z)}$ .

(ii) follows from (i) and from uniqueness of the trace for the algebra  $\mathbb{M}_{\tau(z)}$ .  $\square$

Now, recall that the category of finite dimensional representations of a topological group is equivalent to the category of finite dimensional representations of its Bohr-compactification [16, Prop.16.1.3]. Accordingly, finite dimensional representations of a topological group are classified by their characters, and this shows our claim. *Fourthly*, the topological dimension is stable under conjugation, this is the content of the next result.

**Lemma 6.13.** *Let  $z$  be an irreducible object with finite statistics. Then  $z$  and the conjugate  $\bar{z}$  have the same topological dimension.*

*Proof.* Since  $z$  is irreducible,  $\bar{z}$  is irreducible and  $d(z) = d(\bar{z})$ . Let  $\sigma_z$  and  $\sigma_{\bar{z}}$  be the  $\tau(z)$ - and  $\tau(\bar{z})$ -dimensional representations of  $\pi_1(M, o)$  associated, respectively, with  $z$  and  $\bar{z}$  by Corollary 6.12, and let  $c_{\sigma_z}$  and  $c_{\sigma_{\bar{z}}}$  be the corresponding normalized characters. Given a loop  $\ell$  over  $o$ , using equation (5.11) we have

$$\begin{aligned} d(z) \omega_o^z(z(\ell)) &= r_o^* \bar{y}(o)(z(\ell)) r_o = r_o^* \bar{y}(o)(z(\ell)) \bar{z}(\ell) \bar{z}(\ell)^* r_o \\ &= r_o^* (\bar{z} \times z)(\ell) \bar{z}(\ell)^* r_o = r_o^* \bar{z}(\ell)^* r_o \\ &= r_o^* \varepsilon(z, \bar{z})_o y(o)(\bar{z}(\ell)^*) \varepsilon(\bar{z}, z)_o r_o = \bar{r}_o^* y(o)(\bar{z}(\ell)^*) \bar{r}_o \\ &= d(\bar{z}) \omega_o^{\bar{z}}(\bar{z}(\ell)^*) , \end{aligned}$$

where we have used the relation that  $\bar{r} = \kappa(z) \cdot \varepsilon(\bar{z}, z) \cdot r$  (see [38]). Hence

$$\omega_o^{\bar{z}}(\bar{z}(\ell)) = \omega_o^z(z(\ell))^* , \quad \ell \in \text{Loops}_{K(M)}(o) . \quad (6.11)$$

This relation and (ii) of Corollary 6.12 imply that  $c_{\sigma_{\bar{z}}}$  is equal to the adjoint of  $c_{\sigma_z}$ . Hence  $\sigma_{\bar{z}}$  is equivalent to the conjugated representation of  $\sigma_z$ ; the latter is irreducible and has dimension  $\tau(z)$ .  $\square$

**Remark 6.14.** We do not define the topological dimension for reducible objects. This can be understood by the following observation. If  $z$  is an irreducible 1-cocycle with finite statistics, then the representation of the fundamental group associated with  $z \oplus z$  and  $z$  are equivalent. In fact by Corollary 6.12 we have  $(\mathbb{1}_{\mathcal{H}_0} \otimes \sigma_z) \oplus (\mathbb{1}_{\mathcal{H}_0} \otimes \sigma_z) \cong \mathbb{1}_{\mathcal{H}_0} \otimes \sigma_z$ , because of the infinite multiplicity. Thus, it is not possible to extend the topological dimension to reducible objects by additivity.

Finally, we show the structure of those 1-cocycles whose topological dimension equals the statistical dimension.

**Lemma 6.15.** *Let  $z$  be an irreducible object with finite statistics. Assume that  $\tau(z) = d(z)$ .*

(i) *Then  $\langle z \rangle = u^{\oplus \tau(z)}$  the  $\tau(z)$ -fold direct sum of a simple object  $u$  of  $Z_t^1(\mathcal{R}(K(M)))$ .*

(ii) If  $d(z) = 1$  then  $\tau(z) = 1$ , so  $\langle z \rangle$  is a simple object, while  $\chi_z$  takes values in  $\mathbb{C}$ ; hence

$$z(b) = \chi_z(b) \langle z \rangle(b), \quad b \in \Sigma_1(K(M)) .$$

*Proof.* (i) If  $\tau(z) = d(z)$ , then  $(\langle z \rangle, \langle z \rangle)$  has  $d(z)$  mutually orthogonal projections. Since  $\langle z \rangle$  has statistical dimension equal to  $d(z)$  and since the statistical dimension is additive,  $\langle z \rangle$  is a finite direct sum of  $d(z)$  simple subobjects  $u_i$ . Moreover, all the subobjects of  $\langle z \rangle$  are equivalent. Note that  $\mathcal{R}^z(M, o)$  is type  $I_{\tau(z)}$  factor. So as a linear vector space it has dimension  $\tau(z)^2$ . Observe that if there were a subobject  $u_1$ , to say, which is not equivalent to the other subobjects of  $\langle z \rangle$ , then the algebra  $(\langle z \rangle, \langle z \rangle)_o$  would have, as linear vector space, dimension less than  $(\tau(z) - 1)^2 + 1$ . This leads to a contradiction because  $\mathcal{R}^z(M, o) \subseteq (\langle z \rangle, \langle z \rangle)_o$ . (ii) follows from (i).  $\square$

## 7 Existence and physical interpretation

The first aim of this section is to show that for any irreducible finite dimensional representation of the fundamental group of the spacetime  $M$ , there is an irreducible object with finite statistics carrying, up to infinite multiplicity, this representation. This says that the category  $Z^1(\mathcal{R}_{K(M)})_f$  describes the Bohr-compactification of the fundamental group of the spacetime. The second aim is to show how the topology of spacetime affects the charges associated with  $Z^1(\mathcal{R}_{K(M)})_f$  and to point out the analogy with the Ehrenberg-Siday-Aharonov-Bohm effect.

Recall that any representation of the fundamental group of the spacetime  $M$  defines, up to equivalence, a unique representation of the fundamental group of  $K(M)$ . Thus, let  $\sigma$  be an irreducible  $n$ -dimensional representation of  $\pi_1(K(M), o)$ . Now, pick a simple object  $u$ , possibly  $u = \iota$ , of  $Z^1_t(\mathcal{R}_{K(M)})$  with say Bose-statistics. Define  $z \doteq u^{\oplus n}$  and note that the algebra  $(z, z)_o$  is spatially equivalent to  $\mathbb{1}_{\mathcal{H}_0} \otimes \mathbb{M}_n$ . Let  $U : \mathcal{H}_0 \rightarrow \mathcal{H}_0 \otimes \mathbb{C}^n$  be the unitary operator yielding this equivalence. Define

$$\tilde{\sigma}([\ell]) \doteq U^* \mathbb{1}_{\mathcal{H}_0} \otimes \sigma([\ell]) U, \quad [\ell] \in \pi_1(K(M), o) .$$

Observe that  $\tilde{\sigma}$  is a factor representation taking values in  $(z, z)_o$ . Following [53], by this representation we can define a 1-cocycle  $\varphi_\sigma$  of  $K(M)$  taking values in  $(z, z)_o$ . The representation of  $\pi_1(K(M), o)$  associated with  $\varphi_\sigma$  is equal to  $\tilde{\sigma}$ . Now, define

$$z_\sigma \doteq \varphi_\sigma \rtimes z .$$

By Theorem 6.7  $z_\sigma$  is a 1-cocycle of  $Z^1(\mathcal{R}_{K(M)})$  such that the representation of the fundamental group associated with  $z_\sigma$  is  $\tilde{\sigma}$  (see within the proof the cited theorem). Hence, this fact and Theorem 4.3 prove the existence of topologically non-trivial net representations for the observable net  $\mathcal{A}_{K(M)}$ .

We now show that  $z_\sigma$  is an irreducible object with Bose-statistics and statistical dimension equal to  $n$ . We need a preliminary observation. Since  $\chi_{z_\sigma}$  is equivalent to

$\varphi_\sigma$  and  $\langle z_\sigma \rangle \cong z$  (Theorem 6.7), without loss of generality, from now on we assume  $\chi_{z_\sigma} = \varphi_\sigma$  and  $\langle z_\sigma \rangle = z$ . We now prove that  $z_\sigma$  is irreducible. Assume that there is a non-zero projection  $t \in (z_\sigma, z_\sigma)$  such that  $t \neq \mathbb{1}$ . Clearly  $t \in (z, z)$  and  $t_o$  commutes with  $\varphi_\sigma$ . Moreover,  $U t_o U^*$  is a non-zero projection of  $1_{\mathcal{H}_0} \otimes \mathbb{M}_n$  which is different from the identity of this algebra and commutes with  $\sigma$ . This leads to a contradiction because  $\sigma$  is irreducible. Hence  $z_\sigma$  is irreducible.

Concerning the statistics, one can deduce by the particular form of  $z$ , that there is an isometry  $t \in (v, z^{\otimes n})$  such that  $t \cdot t^* = a_z^n$ , where  $a_z^n$  is the totally antisymmetric projection of  $(z^{\otimes n}, z^{\otimes n})$  defined by the representation  $\varepsilon_z^n$  of the permutation group  $\mathbb{P}(n)$  associated with  $z$ , and  $v$  is a Bosonic simple subobject  $z^{\otimes n}$  (see [18]). Now, we observe that as a consequence of Lemma 6.1 the representation  $\varepsilon_z^n$  equals the representation  $\varepsilon_{z_\sigma}^n$  for any  $n$  (note that the functor  $\mathcal{P}_c$  in Lemma 6.1 acts as the identity on arrows). This, in particular, implies that  $a_{z_\sigma}^n = a_z^n$  and  $\varepsilon(z_\sigma^{\otimes n}, z_\sigma^{\otimes n}) = \varepsilon(z^{\otimes n}, z^{\otimes n})$ .

On these premises, using the defining properties of a permutation symmetry, since  $v$  is a Bosonic simple object, we have  $\mathbb{1} = \varepsilon(v, v) = t^* \otimes t^* \cdot \varepsilon(z^{\otimes n}, z^{\otimes n}) \cdot t \otimes t$ , which is equivalent to saying that  $a_z^n \otimes a_z^n = a_z^n \otimes a_z^n \cdot \varepsilon(z^{\otimes n}, z^{\otimes n})$ . Hence  $a_{z_\sigma}^n \otimes a_{z_\sigma}^n = a_{z_\sigma}^n \otimes a_{z_\sigma}^n \cdot \varepsilon(z_\sigma^{\otimes n}, z_\sigma^{\otimes n})$ . Accordingly, the subobject  $w$  of  $z_\sigma^{\otimes n}$  associated with the projection  $a_{z_\sigma}^n$  is a simple object. This is enough to prove that  $z_\sigma$  has finite statistics [51]. By Proposition 6.9  $z_\sigma$  is Bosonic and has statistical dimension equal to  $n$ .

This leads to the following existence theorem.

**Theorem 7.1.** *Let  $\sigma$  be an irreducible and  $n$ -dimensional representation of the fundamental group of the spacetime  $M$ . Then, there exists a 1-cocycle  $z$  of  $Z^1(\mathcal{R}_{K(M)})$  having the following properties*

- (i)  *$z$  is an irreducible object with finite statistics whose statistical and topological dimension are equal to  $n$ ;*
- (ii) *the representation of the fundamenal group of the spacetime  $M$  associated with  $z$  is equivalent to  $\mathbb{1}_{\mathcal{H}_0} \otimes \sigma$ .*

We draw on some consequences of this result. First, we point out that Theorem 7.1 implies the existence of topologically non-trivial superselection sectors even when the only DHR-sector of the net  $\mathcal{R}_{K(M)}$  is the vacuum, i.e.,  $Z_t^1(\mathcal{R}_{K(M)})$  is formed by finite direct sums of the trivial 1-cocycle  $\iota$ . Secondly, since any finite dimensional representation of  $\pi_1(M, o)$  appears, up to infinite multiplicity, as the representation of  $\pi_1(M, o)$  associated with a 1-cocycle with finite statistics, what we have shown is that the topological content of the category  $Z^1(\mathcal{R}_{K(M)})_f$  is the Bohr-compactification of the fundamental group of  $M$ .

We now turn to explain how the topology of the spacetime  $M$  affects, if not trivial, the charges  $Z^1(\mathcal{R}_{K(M)})_f$ . Consider an irreducible 1-cocycle  $z$  with finite statistics. Fix a point  $x$  of the spacetime, and consider the family  $y_x^z(o)$ , with  $o \in K(x^\perp)$ , of localized transportable endomorphisms of the stalk  $\mathcal{R}(x^\perp)$  associated with  $z$ . According to the interpretation given in DHR analysis,  $y_x^z(o)$  is a representation describing a charge within

the region  $o$ ; the 1-cocycle  $z$  is the transporter of this charge (see Remark 5.2). Then, transport this charge from  $o$  to another region  $\tilde{o}$  along two different paths  $p$  and  $q$  such that the loop  $p * q$  over  $o$  is not homotopic to the trivial loop. Then

$$z(p * \bar{q}) y_x^z(o) = z(p) y_x^z(\tilde{o}) z(\bar{q}) = y_x^z(o) z(p * \bar{q}) \neq y_x^z(o) .$$

This means that, analogously to the Ehrenberg-Siday-Aharonov-Bohm effect [21, 1] the final state differs from the initial one of the unitary  $z(p * \bar{q})$ . The analogy is actually tighter if one thinks of 1-cocycles as flat connections of a principal bundle over the poset  $K(M)$  [49, 50]). Then the difference from the initial to the final state is the parallel transport of the flat connection  $z$  along the loop  $p * \bar{q}$ .

## 8 Comments and outlook

The present paper is, in our opinion, an interesting contribution to the topic of the existence of quantum effects induced by the topology of spacetimes. We have shown the existence of a new class of superselection sectors having well defined charge and topological contents, in the case in which the spacetime is multiply connected. These sectors are sharply localized in the same sense of those discovered by Doplicher, Haag and Roberts [18], but they are affected by the spacetime topology in a way similar to the quantum geometric phases. In a certain sense, the results of the present paper bring some support to the idea, suggested by Ashtekar and Sen [2], that non-trivial topologies may induce the existence of a new kind of particles (see also [31]). We think that the main and the new contribution given by the present paper to that idea resides in the fact that the exposed results are model-independent and based on a few, physically reasonable, assumptions.

We have shown the charge and the topological content of sectors of  $Z^1(\mathcal{R}_{K(M)})$  in a fixed spacetime background  $M$ . As said at the beginning, it would be interesting to understand the locally covariant [11] behavior of these sectors. This may clarify some issues in the analysis of the locally covariant structure of DHR-sectors [12].

We now point out a central question arising from our results.

*Is there an underlying gauge theory giving rise to the charged sectors of  $Z^1(\mathcal{R}_{K(M)})_{\text{f}}$ ?*

We are asking whether it is possible either to provide models of gauge fields giving rise to sectors  $Z^1(\mathcal{R}_{K(M)})_{\text{f}}$  and, in general, whether one is able to reconstruct the fields and the gauge group underlying the charges  $Z^1(\mathcal{R}_{K(M)})_{\text{f}}$ , as Doplicher and Roberts have shown to happen for DHR-sectors and for BF-sectors (the Buchholz-Fredenhagen charges [13]) in Minkowski spacetime [20].

As far as models are concerned, a first positive, but very preliminary step has been provided in [8]: there, it is proven that a massive bosonic quantum field in a 2-dimensional spacetime (the Einstein cylinder) has a non-trivial topological cocycle that gives rise to non-trivial unitary representations of the fundamental group of the circle. Note that, on 2-dimensional Minkowski spacetime, the model does not have any DHR sector of the usual kind besides the vacuum [41]. It is an intriguing question whether

our selection criterion gives something really different from DHR, or new perspectives, in low dimensions. We hope to return elsewhere to this question too.

As far as the abstract construction is concerned, the question about Doplicher-Roberts reconstruction Theorem has, in our opinion, two different answers according to whether the fundamental group of the spacetime is Abelian or not. In the Abelian case, we think that there are no important differences from the scenery suggested by the Doplicher-Roberts reconstruction: there should exist a field net acted upon globally by the gauge group (in this direction goes the example in [8]). Conversely, in the non-Abelian case, we expect the scenery of the Doplicher-Roberts reconstruction to break down. It is reasonable to think that this happens because of the operation of join (6.8) that couples the charge and the topological component of sectors. When the fundamental group is not Abelian the coupling between the topological and the charge component is, in general, *local*: the function describing that coupling within Definition (6.8) depends on 1-simplices. Conversely, this coupling is *global* in the Abelian case because this function reduces to the identity for any 1-simplex (Corollary 6.8).

It is not clear what could be the mathematical structure of the field theory underlying the charges  $Z^1(\mathcal{R}_{K(M)})_f$  in the non-Abelian case. One wonders whether it may resemble, at least in a certain sense, what people describes as topological field theories. Indeed, in that framework, theories of flat connections (for instance, Chern-Simons [25]) are considered, as much as we do at the quantum level by considering 1-cocycles with values operators.

**Acknowledgement.** We gratefully acknowledge discussions with Klaus Fredenhagen and John E. Roberts. We wish to warmly thank Miguel Sánchez for sharing with us his insights in Lorentzian geometry.

## A Proof of Theorem 4.3

We prove the equivalence between  $SC(\mathcal{A}_{K(M)})$  and  $Z^1(\mathcal{R}_{K(M)})$ .

Let  $\{\pi, \psi\}$  and element of  $SC(\mathcal{A}_{K(M)})$ . Given a 1-simplex  $b$ , take a simply connected subspacetime  $N$  such that  $cl(|b|) \subset N$ , and define

$$z^\pi(b) \doteq W_a^{N\partial_0 b} W_a^{N\partial_1 b^*}, \quad cl(a) \subset N, \quad a \perp |b|. \quad (\text{A.1})$$

where  $W^{No}$  is the unitary satisfying the selection criterion (3.8). First of all we prove this definition is well posed. By property 2 of (3.8), if  $\tilde{a} \subseteq a$  then

$$W_{\tilde{a}}^{N\partial_0 b} W_{\tilde{a}}^{N\partial_1 b^*} = W_a^{N\partial_0 b} \psi_{a\tilde{a}} \psi_{a\tilde{a}}^* W_a^{N\partial_1 b^*} = W_a^{N\partial_0 b} W_a^{N\partial_1 b^*}.$$

Since  $cl(|b|) \subset N$  we have that  $|b|$  is a diamond of  $N$ . So the causal complement of  $|b|$ , relatively to  $N$ , is pathwise connected. This and the above identity leads to the independence of the choice of  $a$ . Concerning the independence of the choice of  $N$ , consider a second simply connected subspacetime  $N'$  such that  $cl(|b|) \subseteq N' \cap N$ . By Lemma B.5 there is a diamond  $\mathcal{O}$  and a diamond  $a$  such that  $\mathcal{O} \subseteq N' \cap N$ ,  $cl(|b|), cl(a) \subset$

$\mathcal{O}$  and  $|b| \perp a$ . Note that  $\mathcal{O}$  is a simply connected subspacetime of  $M$ . Property 3 of (3.8) implies that  $W_a^{N\partial_i b} = W_a^{N'\partial_i b} = W_a^{\mathcal{O}\partial_i b}$  for  $i = 0, 1$ . This and the independence of the choice of  $a$ , leads to the independence of the choice of  $N$ . Hence  $z^\pi$  is well defined. Now, the cocycle identity follows from the definition of  $z^\pi$  and from the independence of the choice of  $N$  and  $a$ . In fact given a 2-simplex  $c$  take  $N$  such that  $cl(|c|)$ . By Lemma B.5 there is  $a$  such that  $cl(a) \subset N$  and  $c \perp a$ . Then

$$\begin{aligned} z^\pi(\partial_0 c) z^\pi(\partial_2 c) &= W_a^{N\partial_{00}c} W_a^{N\partial_{10}c*} W_a^{N\partial_{02}c} W_a^{N\partial_{12}c*} \\ &= W_a^{N\partial_{01}c} W_a^{N\partial_{10}c*} W_a^{N\partial_{10}c} W_a^{N\partial_{11}c*} \\ &= W_a^{N\partial_{01}c} W_a^{N\partial_{11}c*} \\ &= z^\pi(\partial_1 c) . \end{aligned}$$

Concerning the locality condition, note that by outer regularity and Haag duality it is easily seen that  $\mathcal{R}(o) = \cap \{\mathcal{R}(\bar{o})' \mid cl(o) \perp cl(\bar{o})\}$  (see for instance [52]). So given a 1-simplex  $b$  pick a 0-simplex  $o$  such that  $cl(|b|) \perp cl(o)$ . By the smoothability argument [6] there is a spacelike Cauchy surface  $\mathcal{C}$  which contains the closure of the bases of the diamonds  $|b|$  and  $o$ . Moreover, since the closure of the bases of  $|b|$  and  $o$  are disjoint there is a simply connected open subset  $G$  of  $\mathcal{C}$  which contains the closure of both the bases. Then, the domain of dependence of  $G$  is a simply connected subspacetime  $N$  which contains  $cl(|b|)$  and  $cl(o)$ . Since (A.1) is independent of the choice of  $a$ , by property 1 of (3.8) we have

$$z^\pi(b) \iota_o(A) = W_o^{N\partial_0 b} W_o^{N\partial_1 b*} \iota_o(A) = \iota_o(A) W_o^{N\partial_0 b} W_o^{N\partial_1 b*} = \iota_o(A) z^\pi(b) ,$$

for any  $A \in \mathcal{A}(o)$ . Since this holds for any  $cl(o) \perp cl(|b|)$ , the above observation implies that  $z^\pi(b) \in \mathcal{A}(|b|)$ . This proves that  $z^\pi \in Z^1(\mathcal{R}_{K(M)})$ .

Consider now  $\{\sigma, \phi\} \in \text{SC}(\mathcal{A}_{K(M)})$  and an arrow  $T \in (\{\pi, \psi\}, \{\sigma, \phi\})$ . For any 0-simplex  $a$  take  $o \perp a$  and  $N$  such that  $cl(a), cl(o) \subset N$ . Define

$$t_a \doteq V_o^{Na} T_o W_o^{Na*} . \quad (\text{A.2})$$

where  $W$  and  $V$  are the unitary associated with  $\{\pi, \psi\}$  and  $\{\sigma, \phi\}$ , respectively, by the selection criterion. As above  $t$  is well defined and independent of the choices of  $o$  and  $N$ . Given a 1-simplex  $b$ , pick  $N$  with  $cl(|b|) \subset N$  and let  $o$  be such that  $|b| \perp o$  and  $cl(o) \subset N$ . Then

$$t_{\partial_0 b} z^\pi(b) = V_o^{N\partial_0 b} T_o W_o^{N\partial_0 b*} W_o^{N\partial_0 b} W_o^{N\partial_1 b*} = V_o^{N\partial_0 b} T_o W_o^{N\partial_1 b*} = z^\sigma(b) t_{\partial_1 b} .$$

One can also easily see that  $t_a \in \mathcal{R}(a)$ , hence  $t \in (z^\pi, z^\sigma)$ .

We now construct the map from net cohomology to net representations satisfying the selection criterion. Given a 1-cocycle  $z \in Z^1(\mathcal{R}_{K(M)})$ , define

$$\begin{aligned} \pi_a^z(A) &\doteq z(q_a) \iota_a(A) z(q_a)^* , & a \in K(M) , A \in \mathcal{A}(a) , \\ \psi_{a,\tilde{a}}^z &\doteq z(a, \tilde{a}) , & \tilde{a} \subseteq a , \end{aligned} \quad (\text{A.3})$$

where  $q_a$  is a path with  $\partial_1 q = a$  and  $\partial_0 q \perp a$ , and  $(a, \tilde{a})$  denotes the 1-simplex such that  $\partial_1(a, \tilde{a}) = \tilde{a}$ ,  $\partial_0(a, \tilde{a}) = |(a, \tilde{a})| = a$ . The independence of the chosen path  $q_a$  is a consequence of (4.5). This also implies that  $\pi_a^z = z(a, \tilde{a}) \pi_a^z z(a, \tilde{a})^* = \psi_{a\tilde{a}}^z \pi_a^z \psi_{a\tilde{a}}^{z*}$  for any  $\tilde{a} \subseteq a$ . Hence the pair  $\{\pi^z, \psi^z\}$  is a net representation. Note that the 1-cocycle  $\zeta^{\pi^z}$ , associated with this net representation by (2.8), is equivalent to  $z$ . In fact, we have

$$\zeta^{\pi^z}(b) = \psi_{|b|\partial_0 b}^z \psi_{|b|\partial_1 b}^{z*} = z(|b|, \partial_0 b)^* z(|b|, \partial_1 b) = z(b) .$$

for any 1-simplex  $b$ . We now prove that this net representation satisfies the selection criterion (3.8). Given  $o \in K(M)$ , let  $N$  be simply connected subspacetime such that  $cl(o) \subset N$ . Given  $a \perp o$ , with  $cl(a) \subset N$ , define

$$W_a^{No} \doteq z(p_{o,a}) , \quad (\text{A.4})$$

where  $p_{o,a}$  is the path from  $a$  to  $o$  whose support has closure contained in  $N$ . Clearly this definition does not depend on  $p$  since  $N$  is simply connected. Furthermore, given a path  $q_a$  as in definition (A.3), by relation (4.4) we have

$$\begin{aligned} W_a^{No} \pi_a^z(A) &= z(p_{o,a}) z(q_a) \iota_a(A) z(q_a)^* \\ &= z(p_{o,a} * q_a) \iota_a(A) z(q_a)^* = \iota_a(A) z(p_{o,a}) \\ &= \iota_a(A) W_a^{No} , \end{aligned}$$

for any  $A \in \mathcal{A}(a)$ , because the endpoints of  $p_{o,a} * q_a$  are causally disjoint from  $a$ . Hence  $\{\pi^z, \psi^z\}$  is a sharp excitation of the reference representation. Finally, given  $t \in (z, \hat{z})$  define

$$T_a \doteq t_a , \quad a \in \Sigma_0(K(M)) . \quad (\text{A.5})$$

Given  $A \in \mathcal{A}(a)$  we have

$$\begin{aligned} T_a \pi_a^z(A) &= t_a z(q_a) \iota_a(A) z(q_a)^* = \hat{z}(q_a) t_{\partial_1 q_a} \iota_a(A) z(q)^* \\ &= \hat{z}(q_a) \iota_a(A) t_{\partial_1 q_a} z(q_a)^* = \hat{z}(q_a) \iota_a(A) \hat{z}(q_a)^* t_a \\ &= \pi_a^{\hat{z}}(A) T_a . \end{aligned}$$

Moreover, it is easily seen that  $T_a \psi_{a,\tilde{a}}^z = \psi_{a,\tilde{a}}^{\hat{z}} T_{\tilde{a}}$ , if  $\tilde{a} \leq a$ . Thus,  $T$  is an intertwiner from  $\{\pi^z, \psi^z\}$  to  $\{\pi^{\hat{z}}, \psi^{\hat{z}}\}$ .

Now, the functor  $F : \text{SC}(\mathcal{A}_{K(M)}) \rightarrow Z^1(\mathcal{B}_{K(M)})$  defined by means of the equations (A.1) and (A.2): given  $\{\pi, \psi\}, \{\sigma, \phi\}$  and  $T \in (\{\pi, \psi\}, \{\sigma, \phi\})$  define

$$\begin{aligned} F(\{\pi, \psi\})(b) &\doteq z^\pi(b) , \quad b \in \Sigma_1(K(M)) , \\ F(T)_a &\doteq V_o^{Na} T_o W_o^{Na*} , \quad a \in \Sigma_0(K(M)) , \end{aligned} \quad (\text{A.6})$$

where  $W$  and  $V$  are the operators associated with  $\{\pi, \psi\}$  and  $\{\sigma, \phi\}$ , respectively, by the selection criterion. The functor  $G : Z^1(\mathcal{B}_{K(M)}) \rightarrow \text{SC}(\mathcal{A}_{K(M)})$  is defined by equations (A.3) and (A.4): given  $z, \hat{z} \in Z^1(\mathcal{B}_{K(M)})$  and  $t \in (z, \hat{z})$ , define

$$\begin{aligned} G(z) &\doteq \{\pi^z, \psi^z\} , \\ G(t) &\doteq t . \end{aligned} \quad (\text{A.7})$$



We first study the composition  $FG$ . Consider a 1-simplex  $b$ . Pick a 0-simplex  $a$  with  $cl(a) \subset N$  and  $a \perp |b|$ . By using (A.4) and (A.1) we have

$$FG(z)(b) = z(p_{\partial_0 b, a}) z(p_{\partial_1 b, a})^* = z(p_{\partial_0 b, a} * \overline{p_{\partial_1 b, a}}) = z(b) ,$$

because  $p_{\partial_0 b, a} * \overline{p_{\partial_1 b, a}}$  is a path from  $\partial_1 b$  to  $\partial_0 b$  and its closure is in  $N$ . Hence it is homotopic to  $b$  because  $N$  is simply connected. Moreover by (A.5) (A.4) (A.2) we have  $FG(t)_a = \hat{z}(p_{a, o}) t_o z(\tilde{p}_{a, o})^* = t_a$  for any 0-simplex  $a$ , because  $p_{a, o}$  and  $\tilde{p}_{a, o}$  are paths from  $o$  to  $a$ . Hence  $FG = \text{id}_{Z^1(\mathcal{R}_{K(M)})}$ . We now show that there is a natural isomorphism from  $GF$  and  $\text{id}_{\text{SC}(\mathcal{A}_{K(M)})}$ . First of all, given  $a$ , take a simply connected region  $N$  which contains  $cl(a)$  and pick  $o, \tilde{a}$  such that  $o \perp a, \tilde{a}$  and  $\tilde{a} \perp o$ . Define

$$S_a(\{\pi, \psi\}) \doteq W_a^{N\tilde{a}*} W_o^{N\tilde{a}} W_o^{Na*}, \quad (\text{A.8})$$

where  $W$  are the unitaries associated with  $\{\pi, \psi\}$  by the selection criterion. By the definitions of  $G$  and  $F$  we have

$$GF(\{\pi, \psi\})_a(A) = F(\{\pi, \psi\})(q_a) \iota_a(A) F(\{\pi, \psi\})(q_a)^* = W_o^{Na} W_o^{N\tilde{a}*} \iota_a(A) W_o^{N\tilde{a}} W_o^{Na*} ,$$

where  $\tilde{a} \perp a$ . In fact, the path  $q_a$  is from  $a$  to  $a^\perp$ . Since, the definition of  $F(\{\pi, \psi\})$  does depend neither on the choice of the path nor on the choice of the 0-simplex in  $a^\perp$ , we have considered a path which lies  $N$  from  $a$  to  $\tilde{a}$ . Finally since  $N$  is simply connected  $F(\{\pi, \psi\})(q_a)$  depends only on the endpoints of  $q_a$ . Thus  $F(\{\pi, \psi\})(q_a) = W_o^{Na} W_o^{N\tilde{a}*}$ . Using this and (A.8) we have

$$\begin{aligned} S_a(\{\pi, \psi\}) GF(\{\pi, \psi\})_a(A) &= S_a(\{\pi, \psi\}) W_o^{Na} W_o^{N\tilde{a}*} \iota_a(A) W_o^{N\tilde{a}} W_o^{Na*} \\ &= W_a^{N\tilde{a}*} \iota_a(A) W_o^{N\tilde{a}} W_o^{Na*} = \pi_a(A) W_a^{N\tilde{a}*} W_o^{N\tilde{a}} W_o^{Na*} \\ &= \pi_a(A) S_a(\{\pi, \psi\}) . \end{aligned}$$

Furthermore, given  $a_1 \subseteq a$  we have

$$\begin{aligned} \psi_{aa_1} S_{a_1}(\{\pi, \psi\}) &= \psi_{aa_1} W_{a_1}^{N\tilde{a}*} W_o^{N\tilde{a}} W_o^{Na_1*} = W_a^{N\tilde{a}*} W_o^{N\tilde{a}} W_o^{Na_1*} \\ &= W_a^{N\tilde{a}*} W_o^{N\tilde{a}} W_o^{Na*} W_o^{Na} W_o^{Na_1*} \\ &= S_a(\pi) \psi_{aa_1}^{z^\pi} . \end{aligned}$$

This proves that  $S(\{\pi, \psi\})$  is a unitary intertwiner from  $GF(\{\pi, \psi\})$  to  $\{\pi, \psi\}$ . Finally, given  $T \in (\{\sigma, \phi\}, \{\pi, \psi\})$  then

$$\begin{aligned} S_a(\{\pi, \psi\}) GF(T)_a &= S_a(\{\pi, \psi\}) F(T)_a \\ &= W_a^{N\tilde{a}*} W_o^{N\tilde{a}} W_o^{Na*} W_o^{Na} T_o V_o^{Na*} = W_a^{N\tilde{a}*} W_o^{N\tilde{a}} T_o V_o^{N\tilde{a}*} V_o^{N\tilde{a}} V_o^{Na*} \\ &= W_a^{N\tilde{a}*} W_a^{N\tilde{a}} T_a V_a^{N\tilde{a}*} V_o^{N\tilde{a}} V_o^{Na*} = T_a V_a^{N\tilde{a}*} V_o^{N\tilde{a}} V_o^{Na*} \\ &= T_a S_a(\{\sigma, \phi\}) , \end{aligned}$$

where the property  $W_o^{N\tilde{a}} T_o V_o^{N\tilde{a}*} = W_a^{N\tilde{a}} T_a V_a^{N\tilde{a}*}$  has been used;  $W$  and  $V$  denote the unitaries associated, respectively, with  $\{\pi, \psi\}$  and  $\{\sigma, \phi\}$  by the selection criterion. This completes the proof of Theorem 4.3.

## B Miscellanea on the causal structure

In this section we provide some results about the causal structure of globally hyperbolic spacetimes. In what follows  $M$  denotes a connected globally hyperbolic spacetime. References for this appendix are the same as those quoted in Section 3.1.

To begin with, we recall that if  $A$  is a closed achronal set then the closure of  $D^\pm(A)$  is the union of those points such that any inextendible past (forward) directed timelike curve starting from these points meets  $A$ .  $D^\pm(A)$  denotes the future (past) domain of dependence of the set  $A$  defined according to the convention of [22, 42].

The first result is due to the courtesy of Miguel Sánchez. We are indebted with him for the nice short proof.

**Lemma B.1** (M. Sánchez). *Let  $\mathcal{C}$  be a spacelike Cauchy surface of  $M$ , and let  $K$  be a nonempty closed subset of  $\mathcal{C}$ . Then  $D^\pm(K)$  is closed.*

*Proof.* Let  $q$  be a point in the boundary of  $D^+(K)$  not included in  $D^+(K)$ . Then there is a past-directed inextendible causal curve through  $q$  which does not cross  $K$ . Nevertheless, it will cross  $\mathcal{C}$  at some point  $p$ . As  $K$  is closed in  $\mathcal{C}$ , a neighborhood  $U$  of  $p$  in  $\mathcal{C}$  does not intersect  $K$ . Easily, there are points in  $U$  chronologically related to  $q$ . ( $p$  lies in  $J^-(q)$ , which is included in the adherence of  $I^-(q)$ ). Let  $p_1$  be one such point. The past directed timelike curve from  $q$  to  $p_1$  cannot intersect  $K$  (as cannot intersect  $\mathcal{C}$  again). This contradicts the fact that  $cl(D^+(K))$  is the set of points  $x$  of  $M$  such that every inextendible timelike curve through  $x$  crosses  $K$ .  $\square$

If  $A \subseteq \mathcal{C}$  let us denote by  $int_{\mathcal{C}}(A)$  the internal part of  $A$  in the relative topology of  $\mathcal{C}$ . Note that as  $\mathcal{C}$  is a closed subset of  $M$ , the closure of  $A$  in the relative topology of  $\mathcal{C}$  coincides with its closure in the topology of  $M$ .

**Lemma B.2.** *Let  $\mathcal{C}$  be a spacelike Cauchy surface of  $M$ , and let  $K$  be a nonempty closed subset of  $\mathcal{C}$ . Assume that  $K \subseteq cl(int_{\mathcal{C}}(K))$ . Then  $cl(D^\pm(int_{\mathcal{C}}(K))) = D^\pm(K)$ .*

*Proof.* As far as the first inclusion is concerned, as  $D^+(int_{\mathcal{C}}(K)) \subset D^+(K)$ , we have  $cl(D^+(int_{\mathcal{C}}(K))) \subseteq D^+(K)$  by Lemma B.1.

For the opposite inclusion, take  $x \in D^+(K)$ . Note that if  $x \in K$  then  $x \in cl(D^+(int_{\mathcal{C}}(K)))$ . In fact as  $K \subseteq cl(int_{\mathcal{C}}(K))$  we have that  $K \subset cl(D^+(int_{\mathcal{C}}(K)))$ . So assume that  $x \in D^+(K) \setminus K$ . Since, by Lemma B.1,  $D^+(K)$  is closed, we have that  $I^-(x) \cap \mathcal{C} \subset K$ . However,  $I^-(x)$  is open in  $M$ , hence  $I^-(x) \cap \mathcal{C}$  is open in the relative topology of  $\mathcal{C}$ ; thus  $I^-(x) \cap \mathcal{C} \subseteq int_{\mathcal{C}}(K)$ . Take a sequence of points  $\{y_n\}$  converging to  $x$  such that  $y_n \in I^-(x) \cap I^+(K)$  for any  $n \in \mathbb{N}$ . Note that  $J^-(y_n) \subset I^-(x)$  because  $y_n \in I^-(x)$ . Hence  $J^-(y_n) \cap \mathcal{C} \subset I^-(x) \cap \mathcal{C} \subseteq int_{\mathcal{C}}(K)$ . Thus  $y_n \in D^+(int_{\mathcal{C}}(K))$  for any  $n$ . Since  $y_n \rightarrow x$ , then  $x \in cl(D^+(int_{\mathcal{C}}(K)))$ . The same argument applies to the past development and the proof is over.  $\square$

**Corollary B.3.** *Let  $\mathcal{C}$  be a spacelike Cauchy surface of  $M$ , and let  $O$  be a nonempty open subset of  $\mathcal{C}$  in the relative topology of  $\mathcal{C}$ . Assume that  $int_{\mathcal{C}}(cl(O)) \subseteq O$ . Then  $cl(D^\pm(O)) = D^\pm(cl(O))$ .*

*Proof.* Note that the closed set  $K \doteq cl(O)$  satisfies the hypothesis of Lemma B.2. In fact  $int_{\mathcal{C}}(cl(O)) = O$ , because  $O$  is open; hence  $cl(int_{\mathcal{C}}(K)) = cl(int_{\mathcal{C}}(cl(O))) = cl(O) = K$ , and the proof follows by Lemma B.2.  $\square$

This result applies to diamonds. Let  $o$  be a diamond whose base  $G$  lies on a spacelike Cauchy surface  $\mathcal{C}$ .  $G$  is an open and relatively compact subset in the relative topology of  $\mathcal{C}$ . Moreover,  $int_{\mathcal{C}}(cl(G)) = G$ . Then as a consequence of the previous result we have

$$cl(o) = cl(D(G)) = D(cl(G)) . \quad (\text{B.1})$$

We now are in a position to prove the following property of diamonds.

**Lemma B.4.** *Let  $o$  be a diamond of  $M$ . Then there is a sequence  $\{o_n, a_n\}$  of pairs of diamonds based on the same Cauchy surface as  $\mathcal{C}$  and satisfying the following properties:*

- (i)  $cl(o_{n+1}) \subset o_n$  for any  $n \in \mathbb{N}$  and  $\cap_n o_n = cl(o)$ ;
- (ii)  $cl(a_n) \subset o_n$  and  $cl(a_n) \perp cl(o_{n+1})$  for any  $n \in \mathbb{N}$ .

*Proof.* The proof is splitted in two parts. In the first step we construct, by induction, a sequence of pairs of “formal” diamonds satisfying all the properties in the statement. By a formal diamond we mean a set satisfying all the properties to be a diamond with the exception that it might be non relatively compact. In the second step we prove that from the above sequence it is possible to extract a subsequence of diamonds.

*First step.* According to the definition of diamonds there is a spacelike Cauchy surface  $\mathcal{C}$ , a chart  $(U, \phi)$  of  $\mathcal{C}$ , and an open ball  $B$  of  $\mathbb{R}^3$  such that  $cl(B) \subset \phi(U)$ ,  $o = D(G)$  where  $G \doteq \phi^{-1}(B)$ . Define  $W_1 \doteq U$ . Since  $cl(B)$  is a proper compact subset of the open  $\phi(W_1)$  the distance  $d$  of  $B$  from  $\mathbb{R}^3 \setminus \phi(W_1)$  is strictly positive. Accordingly, denoting the radius of  $B$  by  $r$ , define  $B_1$  as the open ball having the same centre as  $B$  and radius  $r_1 \doteq r + d/2$ , and let  $L_1$  be an open ball such that  $cl(L_1) \cap cl(B) = \emptyset$  and  $cl(L_1) \subset B_1$ . Observe that if we define  $G_1 \doteq \phi^{-1}(B_1)$  and  $H_1 \doteq \phi^{-1}(L_1)$ , then by construction and by Lemma B.2, we have

$$cl(o), cl(D(H_1)) \subset D(G_1) , \quad cl(o) \perp cl(D(H_1)) , \quad cl(G_1) \subset W_1 . \quad (*)$$

By induction, for  $n \geq 1$ , assume that we have a 3-tuple  $(G_n, H_n, W_n)$  satisfying  $(*)$ . Define  $W_{n+1} \doteq (W_n \setminus cl(H_n)) \cap G_n$ . Applying the above construction with respect to  $W_{n+1}$  we get a pair  $G_{n+1}$  and  $H_{n+1}$ . This procedure leads to a sequence  $\{G_n, H_n, W_n\}$  satisfying  $(*)$  for any  $n$ . Furthermore, by construction we have that  $cl(H_n) \perp cl(G_{n+1})$  and  $cl(G_{n+1}) \subset G_n$  for any  $n$ ; moreover

$$\cap_n G_n = cl(G) . \quad (**)$$

Define  $o_n \doteq D(G_n)$  and  $a_n \doteq D(H_n)$ . Hence we have  $cl(o_{n+1}) \subset o_n$ ;  $cl(a_n) \subset o_n$  and  $cl(a_n) \perp cl(o_{n+1})$  for any  $n$ . We now show that  $\cap_n o_n = cl(o)$ . Clearly  $cl(o) \subseteq \cap_n o_n$ . Let  $x \in \cap_n o_n$ . Since the sets  $G_n$  belong to the same Cauchy surface  $\mathcal{C}$ , there are two

possibilities either  $x \in D^+(G_n)$  for all  $n$ , or  $x \in D^-(G_n)$  for all  $n$ . Assume  $x \in D^+(G_n)$  for all  $n$ . Then,  $J^-(x) \cap \mathcal{C} \subseteq G_n$  for all  $n$ ; hence by  $(**)$   $J^-(x) \cap \mathcal{C} \subseteq cl(G)$ , thus  $x \in D^+(cl(G)) = cl(D^+(G)) \subset cl(o)$ . The same argument applies if  $x \in D^-(G_n)$  for all  $n$ . This proves our claim.

*Second step.* We now prove that the sequence  $\{o_n, a_n\}$  is definitively formed by diamonds of  $M$ . To start with, note that there is a smooth foliation  $F : \Sigma \times \mathbb{R} \rightarrow M$  of the spacetime, by spacelike Cauchy surfaces, such that  $F(\Sigma, 0) = \mathcal{C}$  [6]. To be precise,  $F : \Sigma \times \mathbb{R} \rightarrow M$  is a diffeomorphism such that  $F(\Sigma, t)$  is a spacelike Cauchy surface for any  $t \in \mathbb{R}$  and the curve  $\gamma_y(t) \doteq F(y, t)$  as  $t$  varies in  $\mathbb{R}$ , for a fixed  $y \in \Sigma$ , is an inextendible forward directed timelike curve. The inverse function  $F^{-1}$  is equal to  $(h(x), \tau(x))$  where  $h$  and  $\tau$  are smooth functions from  $M$  to  $\Sigma$  and  $\mathbb{R}$ , respectively. On these grounds, since  $cl(o)$  is compact there are  $t_1, t_2 \in \mathbb{R}$  such that  $t_1 < t_2$  and

$$t_1 < \{\tau(x) \in \mathbb{R} \mid x \in cl(o)\} < t_2 ;$$

in other words  $cl(o)$  is a proper subset of  $F(\Sigma, (t_1, t_2))$ . Define Let  $K^\Sigma \doteq \{h(x) \in \Sigma \mid x \in cl(G_1)\}$ , where  $G_1$  is the first set constructed above.  $K^\Sigma$  is a compact subset of  $\Sigma$ . Note that, by the properties of the foliation, we have  $cl(o_n) = cl(D(G_n))$  is contained in  $F(K^\Sigma, \mathbb{R})$  for any  $n$ . Consider now the cylinder  $F(K^\Sigma, [t_1, t_2])$ . We claim that there exists a  $k$  such that  $cl(o_n) \subseteq F(K^\Sigma, [t_1, t_2])$  for  $n \geq k$ . If it were not so, there should be a sequence of points  $x_n \in cl(o_n) \cap (F(K^\Sigma, t_1) \cup F(K^\Sigma, t_2))$ . So, extract a subsequence  $\{x_{s_n}\}$  which belongs say to  $cl(o_{s_n}) \cap F(K^\Sigma, t_1)$ . This is a compact set because so is  $F(K^\Sigma, t_1)$ . So there is a subsequence  $\{x_{r_{s_n}}\}$  converging to a point  $x$ . Clearly  $x \in F(K^\Sigma, t_1)$ . Moreover  $x \in cl(o)$  because  $x \in cl(o_{r_{s_n}})$  for any  $n$ . This leads to a contradiction because by construction  $cl(o) \subset F(K^\Sigma, (t_1, t_2))$ . Finally, for  $n \geq k$  the sets  $o_n, a_n$  are contained in the compact set  $F(K^\Sigma, [t_1, t_2])$ , so they are relatively compact, and the proof follows.  $\square$

**Lemma B.5.** *Let  $o$  be a diamond of  $M$  and let  $W$  be an open subset of  $M$  such that  $cl(o) \subset W$ . Then there are two diamonds  $\tilde{o}$  and  $a$  based on the same Cauchy surface as  $o$  such that  $cl(\tilde{o}) \subset W$ ,  $cl(o), cl(a) \subset \tilde{o}$  and  $cl(o) \perp cl(a)$ .*

*Proof.* Let  $\{o_n, a_n\}$  be a sequence of diamonds satisfying the properties of Lemma B.4. The idea of the proof is the same as the second step of the proof of Lemma B.4. Briefly, assume that  $cl(o_n) \cap (M \setminus W) \neq \emptyset$  for any  $n$ . Take a sequence of points  $\{x_n\}$  with  $x_n \in cl(o_n) \cap (M \setminus W)$ . Since  $cl(o_n) \cap (M \setminus W) \subset cl(o_1)$ , for any  $n$ , and since the latter is compact, there is a subsequence  $\{x_{s_n}\}$  converging to a point  $x$ . The limit point  $x$  is in  $cl(o) \cap (M \setminus W)$  and this lead to a contradiction. So there is  $k$  such that for  $n \geq k$   $cl(o_n) \subset W$ , and the proof follows by the properties of the sequence  $\{o_n, a_n\}$ .  $\square$

We now show a homotopy deformation result whose proof is very similar to the proof of the Van Kampen theorem about the fundamental groups of topological spaces (see [27]).

**Lemma B.6.** *Let  $M$  be a globally hyperbolic spacetime with dimension  $d \geq 3$ . Let  $o$  be a diamond of  $M$ . If  $\gamma : [0, 1] \rightarrow M$  is a curve with  $\gamma(0), \gamma(1) \in o^\perp$ , then  $\gamma$  is homotopic to a curve  $\tilde{\gamma}$  lying in  $o^\perp$ .*

*Proof.* Let  $x = \gamma(0)$  and  $y = \gamma(1)$ . Since  $o^\perp$  is connected, there is a curve  $\tilde{\gamma} : [0, 1] \rightarrow o^\perp$  such that  $\tilde{\gamma}(0) = y, \tilde{\gamma}(1) = x$ . So, the composition  $\gamma_1 * \gamma$  is a loop over  $x$ .

By the definition of causal complement and since diamonds are relatively compact, we have that  $o^\perp = (cl(o))^\perp$ . So the point  $x$  is causally disjoint from  $cl(o)$ . Let  $G$  be the base of the diamond  $o$ . By Lemma B.5 there is a diamond  $o_1$  whose base  $G_1$  is contained in the same Cauchy surface as  $G$  and such that  $cl(G) \subset G_1$ ,  $G_1 \setminus cl(G)$  is connected and  $cl(o_1) \perp x$ . By [6], there is a Cauchy surface  $\mathcal{C}$  which contains  $G, G_1$  and  $x$  and  $cl(G_1)$  is disjoint from  $x$ . Since any Cauchy surface is a deformation retract of  $M$  the loop  $\gamma_1 * \gamma$  is homotopic to a loop  $\gamma_2$  lying in  $\mathcal{C}$ . The loop  $\gamma_2$  meets  $cl(G)$  (otherwise the proof would be completed), hence it meets  $G_1 \setminus cl(G)$ . Let  $z$  be a point where  $\gamma_1$  meets  $G_1 \setminus cl(G)$ . Since  $\mathcal{C} \setminus cl(G)$  is connected there is a curve  $\tau$ , in  $\mathcal{C} \setminus cl(G)$ , from  $x$  to  $z$ . Then  $\bar{\tau} * \gamma_2 * \tau$  is a loop over  $z$  ( $\bar{\tau}$  is the reverse of  $\tau$ ).

Since  $G_1$  and  $\mathcal{C} \setminus cl(G)$  form an open cover of  $\mathcal{C}$ , we have that  $\bar{\tau} * \gamma_1 * \tau \sim \beta_n * \beta_{n-1} * \dots * \beta_1$  where  $\beta_i$  is a curve, in  $\mathcal{C}$ , contained either in  $G_1$  or  $\mathcal{C} \setminus cl(G)$  (the Lebesgue's covering lemma for  $[0, 1]$ ). However we can assume that any  $\beta_i$  is a loop, over  $z$ , either in  $G_1$  or  $\mathcal{C} \setminus cl(G)$ , because  $G_1 \setminus cl(G)$  is connected. Hence observing that  $G_1$  is contractible, we have that  $\bar{\tau} * \gamma_1 * \tau \sim \beta$ , where  $\beta$  is a loop over  $z$  in  $\mathcal{C} \setminus cl(G)$ . Thus  $\gamma_2 \sim \tau * \beta * \bar{\tau}$ . Clearly  $\tau * \beta * \bar{\tau}$  lies in  $o^\perp$ . Since  $\gamma_2 \sim \gamma_1 * \gamma$ , then  $\gamma \sim \tilde{\gamma}$  where  $\tilde{\gamma} \doteq \bar{\gamma}_1 * \tau * \beta * \bar{\tau}$  is a curve from  $x$  to  $y$  lying in  $o^\perp$ .  $\square$

It is obvious but worth observing that the same result holds if we replace in the statement of this lemma  $o^\perp$  with the causal complement  $x^\perp$  of a point  $x$  of  $M$ .

We now provide a version of this lemma in terms of the poset  $K(M)$ .

**Corollary B.7.** *Let  $M$  be a globally hyperbolic spacetime with dimension  $d \geq 3$ . Let  $o$  be a diamond of  $M$ .*

- (i) *If  $p$  is a path with  $\partial_1 p, \partial_0 p$  in  $o^\perp$ , then  $p$  is homotopic to a path  $q$  whose support is contained in  $o^\perp$ .*
- (ii) *If  $p$  is a path with  $cl(\partial_1 p), cl(\partial_0 p)$  in  $o^\perp$ , then  $p$  is homotopic to a path  $q$  whose support has closure contained in  $o^\perp$ .*

*Proof.* The proof is an easy consequence of Lemma B.6 and [53, Lemma 2.17].  $\square$

As before we observe that the results of this corollary hold if we replace in the statement  $o^\perp$  with the causal complement  $x^\perp$  of a point  $x$  of  $M$ .

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